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UNIVERSITY OF ALBERTA

Martingale Representation, Hedging Policies, and Contingent Claims

BY
DAVID BRYAN COLWELL

A THESIS SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY

EDMONTON, ALBERTA
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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Martingale Representation, Hedging Policies, and Contingent Claims** submitted by **David Bryan Colwell** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in **Statistics and Applied Probability**.

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To my Parents and Michele

ABSTRACT

The representation of martingales as stochastic integrals plays a central role in the modern theory of option pricing and hedging, particularly when the martingale representation takes place with respect to an equivalent probability measure. Since the corresponding integrands immediately lead to hedging strategies, it is of considerable interest to find explicit expressions for these integrands.

In this paper, the underlying process, ξ (the stock price) is the solution to a stochastic differential equation with Markov coefficients, as is the Girsanov exponential corresponding to the equivalent measure. In the case where ξ is a continuous diffusion, we provide an explicit expression for the integrands in the martingale representation under the equivalent measure and then apply this result to the problem of finding a riskless hedging strategy corresponding to a contingent claim. We next derive martingale representation results in the cases where ξ is a pure jump process or a diffusion with jumps. Finally, we use these representations to produce an explicit expression for the local risk minimizing hedge portfolio generating a contingent claim when the stock price, ξ, is not necessarily continuous. In the process of deriving this hedge portfolio, we also produce the appropriate equivalent probability measure, which is just the minimal martingale measure of Schweizer. Thus, we provide another motivation for the minimal martingale measure. Conditions for the uniqueness of this pricing measure are also given.

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Chapter 1

MARTINGALE REPRESENTATION AND HEDGING POLICIES

1. Introduction

In the modern theory of option pricing and hedging, the representation of martingales as stochastic integrals plays a central role. Since the corresponding integrands immediately lead to hedging strategies, it is of considerable interest to find explicit expressions for these integrands.

The martingale representation result and its background is fully described in the paper of Ocone [24], where the problem is discussed using methods of the Malliavin calculus and weak differentiability in certain Sobolev spaces. In a recent paper by Ocone and Karatzas [25] the representation result of [24] is extended and then applied to determine optimal portfolios and hedging strategies.

In the Markov case elementary methods, which do not use the Malliavin calculus in function space, are employed by Elliott and Kohlmann in [9] and [10] to determine the integrands in certain stochastic integrals. Indeed, all that is used is the Markov property and the Itô differentiation rule. Similar results are also obtained in [25].

This chapter extends the representation result of [10] to the situation where the martingale representation takes place with respect to an equivalent measure whose Girsanov exponential is defined in terms of a Markov integrand. The motivation for the Girsanov measure transformation is developed by Harrison, Kreps

and Pliska [15], [16]. A Markov Girsanov transform clearly introduces a new integrand in the martingale representation, and this is made explicit in Theorem 3.1. It is possible this result could be derived as a corollary of the general result of Haussman and Ocone, see [24]. However any such relation is certainly not transparent and our proof, again in the Markov case, is simple and direct.

The application of our martingale representation result to option pricing is described in Section 4. Stock price dynamics which give explicit, closed form expressions for hedging policies appear hard to find. However, in Section 5 we show how our result gives the hedging policy in the well known Black–Scholes case, [2], of log-normal prices with constant drift and variance.

This chapter comprises a paper by D.B. Colwell, R.J. Elliott, and P.E. Kopp that is to appear in *Stochastic Processes and their Applications*.

2. Dynamics

Suppose $w = (w^1, \ldots, w^m)$ is an m-dimensional Brownian motion defined for $t \geq 0$ on a probability space $(\Omega, \mathcal{F}, P)$. Consider the d-dimensional stochastic differential equation

$$dx_t = \mu(t, x_t)dt + \sigma(t, x_t)dw_t \tag{2.1}$$

for $t \geq 0$, where $\mu : [0, \infty) \times R^d \to R^d$ and $\sigma : [0, \infty) \times R^d \to R^d \otimes R^m$ are measurable functions which are three times differentiable in x, and which, together with their derivatives, have linear growth in x. Write $\xi_{s,t}(x)$ for the solution of (2.1) for $t \geq s$ having initial condition $\xi_{s,s}(x) = x$. Then from the results of Bismut

[3] or Kunita [19] there is a set $N \subset \Omega$ of measure zero such that for $\omega \notin N$ there is a version of $\xi_{s,t}(x)$ which is twice differentiable in x and continuous in t and s.

Write $D_{s,t}(x) = \dfrac{\partial \xi_{s,t}(x)}{\partial x}$ for the Jacobian of the map $x \to \xi_{s,t}(x)$; then it is known that D is the solution of the linearized equation

$$dD_{s,t}(x) = \mu_x(t, x_t) D_{s,t}(x) dt + \sigma_x(t, x_t) D_{s,t}(x) dw_t \tag{2.2}$$

with initial condition $D_{s,s}(x) = I$, the $d \times d$ identity matrix. The inverse $D_{s,t}^{-1}(x)$ exists; see [3].

Suppose $g : [0, \infty) \times R^d \to R^m$ satisfies similar conditions to those of μ and define the (scalar) exponential $M_{s,t}(x)$ by

$$\begin{aligned} M_{s,t}(x) &= 1 + \int_s^t M_{s,r}(x) g(r, \xi_{s,r}(x)) \cdot dw_r \\ &= 1 + \int_s^t dw_r^* \cdot g(r, \xi_{s,r}(x)) M_{s,r}(x), \end{aligned} \tag{2.3}$$

where * denotes adjoint and $\cdot$ inner product in R^d.

Write $\{F_t\}$ for the right continuous complete family of σ-fields generated by w. If, for example, g further satisfies a linear growth condition

$$|g(t, x)| \leq K(1 + |x|)$$

a new probability measure $\widetilde{P}$ can be defined by putting

$$\frac{d\widetilde{P}}{dP}\Big|_{F_t} = M_{0,t}(x_0).$$

Girsanov's theorem then implies that $\widetilde{w}$ is an $\{F_t\}$ Brownian motion under $\widetilde{P}$ where

$$d\widetilde{w}_t = dw_t - g(t, \xi_{0,t}(x_0)) dt. \tag{2.4}$$

Let $c : R^d \to R$ be a C^2 function which, together with its derivatives, has linear growth, and for $0 \le t \le T$ consider the $\widetilde{P}$ martingale

$$N_t = \widetilde{E}[c(\xi_{0,T}(x_0)) \mid F_t].$$

Then from, for example, Theorem 16.22 of [7] N_t has a representation for $0 \le t \le T$ as

$$N_t = N_0 + \int_0^t \gamma_s d\widetilde{w}_s, \tag{2.5}$$

where γ is an $\{F_t\}$ predictable process such that

$$\int_0^T \widetilde{E}|\gamma_s|^2 ds < \infty.$$

3. Martingale Representation

THEOREM 3.1.

$$\gamma_t = \widetilde{E}\Big[\int_t^T d\widetilde{w}_r^* \cdot g_\xi(r, \xi_{0,r}(x_0)) D_{0,r}(x_0) \cdot c(\xi_{0,T}(x_0)) + c_\xi(\xi_{0,T}(x_0)) D_{0,T}(x_0) \mid F_t\Big] D_{0,t}^{-1}(x_0) \cdot \sigma(t, \xi_{0,t}(x_0)).$$

Proof: For $0 \le t \le T$ write $x = \xi_{0,t}(x_0)$. By the semigroup property of stochastic flows, which follows from the uniqueness of solutions of (2.1),

$$\xi_{0,T}(x_0) = \xi_{t,T}(\xi_{0,t}(x_0)) = \xi_{t,T}(x). \tag{3.1}$$

Differentiating (3.1) we see

$$D_{0,T}(x_0) = D_{t,T}(x) D_{0,t}(x_0). \tag{3.2}$$

Furthermore,

$$M_{0,T}(x_0) = M_{0,t}(x_0)M_{t,T}(x). \tag{3.3}$$

For $y \in R^d$ define $\widetilde{V}(t,y) = E[M_{t,T}(y)c(\xi_{t,T}(y))]$, and consider the martingale

$$\begin{aligned} N_t &= \widetilde{E}[c(\xi_{0,T}(x_0)) \mid F_t] \\ &= \frac{E[M_{0,T}(x_0)c(\xi_{0,T}(x_0)) \mid F_t]}{E[M_{0,T}(x_0) \mid F_t]} \\ &= E[M_{t,T}(x)c(\xi_{t,T}(x)) \mid F_t] \\ &= E[M_{t,T}(x)c(\xi_{t,T}(x))], \quad \text{by the Markov property.} \end{aligned}$$

Then from Lemma 14.18 of [7]

$$N_t = \widetilde{V}(t,x).$$

We noted above that $\xi_{t,T}(x)$ is twice differentiable in x; the differentiability of $E[M_{t,T}(x)c(\xi_{t,T}(x))]$ in t can be established by writing the backward equation for $(M_{t,T}(x), \xi_{t,T}(x))$ as in [19]. A direct proof for the jump case is given in Lemma 2.2.6.

Under $\widetilde{P}$, $\xi_{0,t}(x_0)$ is given by the equation

$$\begin{aligned} \xi_{0,t}(x_0) = x_0 &+ \int_0^t \big(\mu(s,\xi_{0,s}(x_0)) + \sigma \cdot g(s,\xi_{0,s}(x_0))\big)ds \\ &+ \int_0^t \sigma(s,\xi_{0,s}(x_0))d\widetilde{w}_s. \end{aligned} \tag{3.4}$$

If $\widetilde{V}(t,\xi_{0,t}(x_0))$ is expanded by the Itô rule we see

$$\begin{aligned} \widetilde{V}(t,x) = \widetilde{V}(t,\xi_{0,t}(x_0)) &= N_t \\ &= \widetilde{V}(0,x_0) + \int_0^t \Big(\frac{\partial \widetilde{V}}{\partial t}(s,\xi_{0,s}(x_0)) + L\widetilde{V}(s,\xi_{0,s}(x_0))\Big)ds \\ &+ \int_0^t \frac{\partial \widetilde{V}}{\partial x}(s,\xi_{0,s}(x_0))\sigma(s,\xi_{0,s}(x_0))d\widetilde{w}_s. \end{aligned} \tag{3.5}$$

Here

$$L = \sum_{i=1}^{d} \Big(\mu^i + \sum_{j=1}^{m} \sigma_{ij} g^j\Big) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

where $a(t, x_i) = (a_{ij}(t, x_t))$ is the matrix $\sigma\sigma^*$. Now N_t is a special semimartingale, so the decompositions (2.5) and (3.5) must be the same. As there is no bounded variation term in (2.5) we see immediately, similarly to [11], that

$$\frac{\partial \widetilde{V}}{\partial t}(s, \xi_{0,s}(x_0)) + L\widetilde{V}(s, \xi_{0,s}(x_0)) = 0$$

with $\widetilde{V}(T, x) = c(x)$. Also

$$\gamma_s = \frac{\partial \widetilde{V}}{\partial x}(s, \xi_{0,s}(x_0))\sigma(s, \xi_{0,s}(x_0)).$$

However, $\xi_{t,T}(x) = \xi_{0,T}(x_0)$ so, from the differentiability and linear growth of g:

$$\frac{\partial \widetilde{V}(t,x)}{\partial x} = E\Big[\frac{\partial M_{t,T}(x)}{\partial x} c(\xi_{0,T}(x_0)) + M_{t,T}(x) \frac{\partial c}{\partial x}(\xi_{t,T}(x))\Big].$$

Again using the existence of solutions of stochastic differential equations which are differentiable in their initial conditions, we have from (2.3)

$$\frac{\partial M_{t,T}(x)}{\partial x} = \int_t^T dw_r^* \cdot g(r, \xi_{t,r}(x)) \frac{\partial M_{t,r}(x)}{\partial x} dr + \int_t^T dw_r^* \cdot g_\xi(r, \xi_{t,r}(x)) \frac{\partial \xi_{t,r}(x)}{\partial x} M_{t,r}(x). \tag{3.6}$$

However, we can solve (3.6) by variation of constants and obtain

$$\frac{\partial M_{t,T}(x)}{\partial x} = M_{t,T}(x) \cdot \int_t^T d\widetilde{w}_r^* \cdot g_\xi(r, \xi_{t,r}(x)) D_{t,r}(x). \tag{3.7}$$

The result can be verified by differentiation, because (3.6) has a unique solution. Therefore, with $x = \xi_{0,t}(x_0)$,

$$\begin{aligned}\frac{\partial \widetilde{V}(t,x)}{\partial x} &= E\Big[M_{t,T}(x)\Big\{\int_t^T d\widetilde{w}_r^* \cdot g_\xi(r,\xi_{t,r}(x))D_{t,r}(x)\cdot c(\xi_{0,T}(x_0)) \\ &\qquad + c_\xi(\xi_{t,T}(x))D_{t,T}(x)\Big\}\Big] \\ &= \widetilde{E}\Big[\int_t^T d\widetilde{w}_r^* \cdot g_\xi(r,\xi_{0,r}(x_0))D_{0,r}(x_0)\cdot c(\xi_{0,T}(x_0)) \\ &\qquad + c_\xi(\xi_{0,T}(x_0))D_{0,T}(x_0)\mid F_t\Big]D_{0,t}^{-1}(x_0),\end{aligned}$$

and the result follows.

Remark 3.2: The result extends immediately to functions for which a generalized Itô formula holds; this class includes convex functions and differences of two convex functions. See Karatzas and Shreve [18].

4. Hedging Portfolios

It is shown in Harrison and Pliska [16] that hedging policies arise from a martingale representation under an equivalent measure. Consequently, we give an application of Theorem 3.1 in this section.

Consider a vector of d stocks

$$S = (S^1, \ldots, S^d)'$$

whose prices are described by a system of stochastic differential equations of the form

$$dS_t^i = S_t^i\Big(\mu^i(t,S_t)dt + \sum_{j=1}^d \lambda_{ij}(t,S_t)dw_t^j\Big).$$

Models of this kind are usual in finance. When the μ^i and λ_{ij} are constant we have the familiar log-normal stock price. For economic reasons, so that the claim is attainable, see [16], the number of sources of noise, that is the dimension of the Brownian motion w, is taken equal to the number of stocks. $\Lambda_t = \Lambda(t,S) = (\lambda_{ij}(t,S))$ is, therefore, a $d \times d$ matrix. We suppose Λ is non-singular, three times differentiable in S, and that $\Lambda^{-1}(t,S)$ and all derivatives of Λ have at most linear growth in S. Writing $\mu(t,S) = (\mu^1(t,S),\ldots,\mu^d(t,S))'$ we also suppose μ is three times differentiable in S with all derivatives having at most linear growth in S.

We suppose the stocks pay no dividends. However, suppose there is a bond S_t^0 with a fixed interest rate r, so $S_t^0 = e^{rt}$. The discounted stock price vector $\xi_t = (\xi_t^1,\ldots,\xi^d)'$ is then $\xi_t := e^{-rt}S_t$ so

$$d\xi_t^i = \xi_t^i\Big((\mu^i(t,e^{rt}\xi_t) - r)dt + \sum_{j=1}^{d}\lambda_{ij}(t,e^{rt}\xi_t)dw_t^j\Big). \tag{4.1}$$

Writing

$$\Delta_t = \Delta(t,\xi_t) = \begin{pmatrix} \xi_t^1 & & 0 \\ & \ddots & \\ 0 & & \xi_t^d \end{pmatrix}$$

and $\rho = (r,r,\ldots,r)'$ equation (4.1) can be written

$$d\xi_t = \Delta_t((\mu - \rho)dt + \Lambda_t dw). \tag{4.2}$$

As in Section 2, there is a flow of diffeomorphisms $x \to \xi_{s,t}(x)$ associated with this system, and their non-singular Jacobians $D_{s,t}$.

In the terminology of Harrison and Pliska, [16], the return process $Y_t = (Y_t^1,\ldots,Y_t^d)$ is here given by

$$dY = (\mu - \rho)dt + \Lambda dw. \tag{4.3}$$

The drift term in (4.3) can be removed by applying the Girsanov change of measure. Write $\eta(t,S) = \Lambda(t,S)^{-1}(\mu(t,S) - \rho)$ and define the martingale M by

$$M_t = 1 - \int_0^t M_s \eta(s,S)' dw_s.$$

Then

$$M_t = \exp\Big(- \int_0^t \eta' dw_s - \frac{1}{2}\int_0^t |\eta_s|^2 ds \Big)$$

is the Radon–Nikodym derivative of a probability measure $\widetilde{P}$. Furthermore, under $\widetilde{P}$, $\widetilde{w}_t = w_t + \int_0^t \eta(s,S)' ds$ is a standard Brownian motion. Consequently, under $\widetilde{P}$

$$dY_t = \Lambda_t d\widetilde{w}_t$$

and

$$d\xi_t = \Delta_t \Lambda_t d\widetilde{w}_t. \tag{4.4}$$

Therefore, the discounted stock price process ξ is a martingale under $\widetilde{P}$ so $\widetilde{P}$ is a 'risk-neutral' measure.

Consider a function $\overline{\psi} : R^d \to R$, where $\overline{\psi}$ is twice differentiable and $\overline{\psi}$ and $\overline{\psi}_x$ are of at most linear growth in x. For some future time $T > t$ we shall be interested in finding the current price (i.e., current valuation at time t), of a contingent claim of the form $\overline{\psi}(S_T)$. It is convenient to work with the discounted stock price, so we consider equivalently the current value of

$$\psi(\xi_T) := \overline{\psi}(e^{rT}\xi_T).$$

ψ has linear growth, so we may define the square integrable $\widetilde{P}$ martingale N by

$$N_t = \widetilde{E}[\psi(\xi_T) \mid F_t], \qquad 0 \le t \le T.$$

As in Harrison and Pliska, [16], if we can express N in the form

$$N_t = \widetilde{E}[\psi(\xi_T)] + \int_0^t \phi(s)' d\xi_s$$

then $\phi = (\phi^1, \ldots, \phi^d)'$ is a hedge portfolio that generates the contingent claim. However, we can apply Theorem 3.1 to derive immediately:

THEOREM 4.1.

$$N_t = \widetilde{E}[\psi(\xi_T)] + \int_0^t \phi(s)' d\xi_s$$

where

$$\phi(s) = \widetilde{E}\Big[\int_s^T \eta_\xi(u, e^{ru}\xi_{0,u}(x_0)) D_{0,u}(x_0) d\widetilde{w}_u \cdot \psi(\xi_{0,T}(x_0)) + \psi_\xi(\xi_{0,T}(x_0)) D_{0,T}(x_0) \mid F_s\Big] D_{0,s}^{-1}(x_0).$$

Proof: From Theorem 3.1, under measure $\widetilde{P}$

$$N_t = \widetilde{E}[\psi(\xi_T)] + \int_0^t \gamma_s d\widetilde{w}_s$$

where

$$\gamma_s = \widetilde{E}\Big[\int_s^T \eta_\xi D_{0,u}(x_0) d\widetilde{w}_u \cdot \psi(\xi_{0,T}(x_0)) + \psi_\xi(\xi_{0,T}(x_0)) D_{0,T}(x_0) \mid F_s\Big] D_{0,s}^{-1}(x_0) \Delta(\xi_{0,s}(x_0)) \Lambda_s.$$

Because $d\xi_t = \Delta_t \Lambda_t d\widetilde{w}_t$, $\phi(s)$ has the stated form.

Remarks 4.2: Note that if η is not a function of ξ, (which is certainly the situation in the usual log-normal case where μ and Λ are constant), η_ξ is zero and the first term in ϕ vanishes.

The bond component ϕ^0 in the portfolio is given by

$$\phi_t^0 = N_t - \sum_{i=1}^{d} \phi_t^i \xi_t^i, \qquad 0 \le t \le T$$

and N_t is the price associated with the contingent claim at time t.

5. Examples

Stock price dynamics for which the hedging policy ϕ can be evaluated in closed form appear hard to find. However, if we consider a vector of log-normal stock prices we can re-derive the Black-Scholes results. Suppose, therefore, that the vector of stock prices $S = (S^1, \ldots, S^d)'$ evolves according to the equations

$$dS_t^i = S_t^i\Big(\mu^i dt + \sum_{j=1}^{d} \lambda_{ij} dw_t^j\Big) \tag{5.1}$$

where $\mu = (\mu^1, \ldots, \mu^d)$ and $\Lambda = (\lambda_{ij})$, are constant. The discounted stock price ξ is then given by (4.2).

Consider a contingent claim which consists of d European call options with expiry dates $T_1 \le T_2 \le \cdots \le T_d$ and exercise prices $c_1, \ldots, c_d$, respectively. Then

$$\psi(T_1, \ldots, T_d) = \sum_{k=1}^{d} \psi^k(\xi_{0,T_k}(x_0)) = \sum_{k=1}^{d} (\xi_{0,T_k}^k(x_0) - c_k e^{-rT_k})^+.$$

The ψ^k are convex functions, so applying the generalized Itô differentiation rule of [18] Theorem 3.1 is valid as noted in Remark 3.2 with

$$\psi_\xi^k = (0, \ldots, 0, I\{\xi_{0,T_k}^k > c_k e^{-rT_k}\}, 0, \ldots, 0).$$

From (5.1) we see that the Jacobian $D_{0,t}$ is just the diagonal matrix

$$D_{0,t} = \begin{bmatrix} \exp\Big\{ \sum_{j=1}^{d} \lambda_{1j}\widetilde{w}_t^j - \frac{1}{2}\, a_{11}t \Big\} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \exp\Big\{ \sum_{j=1}^{d} \lambda_{dj}\widetilde{w}_t^j - \frac{1}{2}\, a_{dd}t \Big\} \end{bmatrix}$$

and its inverse is

$$D_{0,t}^{-1} = \begin{bmatrix} \exp\Big\{ -\Big(\sum_{j=1}^{d} \lambda_{1j}\widetilde{w}_t^j - \frac{1}{2}\, a_{11}t \Big)\Big\} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \exp\Big\{ -\Big(\sum_{j=1}^{d} \lambda_{dj}\widetilde{w}_t^j - \frac{1}{2}\, a_{dd}t \Big)\Big\} \end{bmatrix}.$$

(The explicit, exponential form of the solution shows $D_{0,t}$ is independent of x_0.)

Thus, the trading strategy ϕ_k that generates the contingent claim $\psi^k(\xi_{T_k})$ is

$$\begin{aligned} \phi_k(s)' &= \widetilde{E}[\psi_\xi^k(\xi_{0,T_k}(x_0))D_{0,T_k} \mid \mathcal{F}_s]D_{0,s}^{-1} \\ &= \Big(0,\dots,0,\widetilde{E}\Big[I\{\xi_{0,T_k} > c_k e^{-rT_k}\} \exp\Big\{ \sum_{j=1}^{d} \lambda_{kj}(\widetilde{w}_{T_k}^j - \widetilde{w}_s^j) \\ &\qquad - \frac{1}{2}\, a_{kk}(T_k - s)\Big\} \mid \mathcal{F}_s\Big], 0, \dots, 0\Big), \end{aligned}$$

for $0 \le s \le T_k$. Note that $\phi_k(s) = 0$ for $s > T_k$, i.e., $\phi_k(s)$ stops at T_k. However, from (5.1),

$$\xi_{0,T_k}^k(x_0) = x_0^k \exp\Big\{ \sum_{j=1}^{d} \lambda_{kj}\widetilde{w}_{T_k}^j - \frac{1}{2}\, a_{kk}T_k \Big\} > c_k e^{-rT_k}$$

iff

$$\sum_{j=1}^{d} \lambda_{kj}\widetilde{w}_{T_k}^j > \log\Big(\frac{c_k}{x_0^k}\Big) + \Big(\frac{1}{2}\, a_{kk} - r\Big)T_k = \alpha_k, \quad \text{say;} \tag{5.2}$$

that is, iff

$$\sum_{j=1}^{d} \lambda_{kj}(\tilde{w}^j_{T_k} - \tilde{w}^j_s) > \alpha_k - \sum_{j=1}^{d} \lambda_{kj}\tilde{w}^j_s.$$

Now, under $\tilde{P}$, $\sum_{j=1}^{d} \lambda_{kj}(\tilde{w}^j_{T_k} - \tilde{w}^j_s)$ is normally distributed with mean zero, variance $a_{kk}(T_k - s)$, and is independent of $\mathcal{F}_s$. Therefore, the nonzero component of $\phi_k(s)$ is

$$\begin{aligned}
&\int_{\alpha_k - \sum_{j=1}^{d} \lambda_{kj}\tilde{w}^j_s}^{\infty} \exp\{x - \tfrac{1}{2}\, a_{kk}(T_k - s)\} \exp\left\{\frac{-x^2}{2a_{kk}(T_k - s)}\right\} \frac{dx}{\sqrt{2\pi a_{kk}(T_k - s)}} \\
&= \int_{\alpha_k - \sum_{j=1}^{d} \lambda_{kj}\tilde{w}^j_s}^{\infty} \exp\left\{\frac{-[x - a_{kk}(T_k - s)]^2}{2a_{kk}(T_k - s)}\right\} \frac{dx}{\sqrt{2\pi a_{kk}(T_k - s)}} \\
&= \int_{\frac{\alpha_k - \sum \lambda_{kj}\tilde{w}^j_s - a_{kk}(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} \\
&= \Phi\Big(\frac{-\alpha_k + \sum \lambda_{kj}\tilde{w}^j_s + a_{kk}(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}\Big).
\end{aligned}$$

Again from (1), $\sum_{j=1}^{d} \lambda_{kj}\tilde{w}^j_s = \log\left(\frac{\xi^k_{0,s}(x_0)}{x^k_0}\right) + \frac{1}{2}\, a_{kk}s$, which together with (5.2) gives

$$\phi_k(s) = \left(0, \ldots, 0, \Phi\left(\frac{\log\left(\frac{\xi^k_{0,s}(x_0)}{c_k}\right) - \frac{1}{2}\, a_{kk}(T_k - s) + rT_k}{\sqrt{a_{kk}(T_k - s)}}\right), 0, \ldots, 0\right)',$$

or, in terms of the (nondiscounted) price S^k_s,

$$\phi_k(s) = \left(0, \ldots, 0, \Phi\left(\frac{\log\left(\frac{S^k_s}{c_k}\right) - (\frac{1}{2}\, a_{kk} - r)(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}\right), 0, \ldots, 0\right)', \tag{5.3}$$

$0 \le s \le T_k$. Therefore, the trading strategy ϕ generating $\psi(T_1, \ldots, T_k) = \sum_{k=1}^{d} \psi^k(\xi_{T_k})$ can be written, with a slight abuse of notation, as $\phi(s) = (\phi_1(s), \ldots, \phi_d(s))'$, where

$$\phi_k(s) = I_{\{s \le T_k\}} \Phi\left(\frac{\log\left(\frac{S^k_s}{c_k}\right) - (\frac{1}{2}\, a_{kk} - r)(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}\right). \tag{5.4}$$

Finally, we calculate the price of the claim $\widetilde{E}[\psi(T_1,\ldots,T_d)] = \sum_{k=1}^{d} \widetilde{E}[\psi^k(\xi_{T_k})]$ similarly:

$$\sum_{k=1}^{d} \widetilde{E}[\psi^k(\xi_{T_k})] = \sum_{k=1}^{d} \widetilde{E}(\xi^k_{T_k} - c_k e^{-rT_k})^+$$

$$= \sum_{k=1}^{d} \widetilde{E}\Big[I\Big\{\sum_{j=1}^{d} \lambda_{kj}\tilde{w}^j_{T_k} > \alpha_k\Big\}\Big(Z_0 \exp\Big\{\sum_{j=1}^{d} \lambda_{kj}\tilde{w}^j_{T_k} - \frac{1}{2}\, a_{kk}T_k\Big\} - c_k e^{-rT_k}\Big)\Big]$$

$$= \sum_{k=1}^{d} S^k_0 \Phi\left(\frac{\log\left(\frac{S^k_0}{c_k}\right) + (\frac{1}{2}\, a_{kk} + r)T_k}{\sqrt{a_{kk}T_k}}\right)$$

$$- c_k e^{-rT_k} \Phi\left(\frac{\log\left(\frac{S^k_0}{c_k}\right) + (\frac{1}{2}\, a_{kk} + r)T_k}{\sqrt{a_{kk}T_k}} - \sqrt{a_{kk}T_k}\right)$$

(where we have used $\xi^k_0 = S^k_0,\ k = 1,\ldots,d$). When $d = 1$ the above result reduces to the well-known Black–Scholes formula.

Chapter 2

PURE JUMP PROCESSES

1. Introduction

In Chapter 1, we derived a martingale representation result for a continuous diffusion under a change of measure. We then applied that theorem to the problem of hedging against contingent claims. Our goal is to extend this result to the case where the stock price is described by a diffusion with jumps. In this chapter, we begin this process of generalizing our earlier work by considering a *pure jump process*, that is, a semimartingale whose martingale part is a purely discontinous martingale. Again, we use the methods of stochastic flows, and produce a martingale representation result under a change of measure.

Because the noise in our pure jump process is given by a random measure, we begin with some definitions and fundamental results for random measures. For a thorough discussion, with proofs, we refer the reader to Jacod and Shiryaev [17] (particularly Chapter II).

Suppose a stochastic basis $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is given.

Definition 1.1. A *random measure* on $\mathbf{R}_+ \times \mathbf{R}$ is a family $\nu = (\nu(\omega; dt, dy) : \omega \in \Omega)$ of nonnegative measures on $(\mathbf{R}_+ \times \mathbf{R}, \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{B}(\mathbf{R}))$ such that $\nu(\omega; \{0\} \times \mathbf{R}) = 0$ for all $\omega \in \Omega$ (that is, we assume there are no jumps at $t = 0$). Here $\mathcal{B}(\mathbf{R})$ denotes the Borel sets on $\mathbf{R}$. Note that our "auxiliary space" is just $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$.

Notation 1.2. Let $\mathcal{O}$ (resp. $\mathcal{P}$) denote the optional (resp. predictable) σ-fields on $\Omega \times \mathbf{R}_+$. Write $\widetilde{\Omega} := \Omega \times \mathbf{R}_+ \times \mathbf{R}$, $\widetilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{B}(\mathbf{R})$ and $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbf{R})$. Suppose

the function $W : \Omega \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is $\tilde{\mathcal{O}}$-measurable. Then for any ω, the mapping $(t,x) \mapsto W(\omega,t,x)$ is $\mathcal{B}(\mathbf{R}_+) \otimes \mathcal{B}(\mathbf{R})$-measurable, and so we can define integration by

$$\int_0^t \int_{\mathbf{R}} W(\omega,s,y)\nu(\omega;ds,dy) = \int_{[0,t]\times\mathbf{R}} W(\omega,s,y)\nu(w;ds,dy)$$

if $\int_0^t \int_{\mathbf{R}} |W| d\nu < \infty$.

Definition 1.3. (a) A random measure ν is *optional* (*predictable*) if for any $\tilde{\mathcal{O}}$ ($\tilde{\mathcal{P}}$)-measurable function W, the process

$$\int_0^t \int_{\mathbf{R}} W(\omega,s,y)\nu(\omega;ds,dy)$$

is optional (predictable).

(b) An optional random measure, ν, is *$\tilde{\mathcal{P}}$-σ-finite* if there is a $\tilde{\mathcal{P}}$-measurable partition (A_n) of $\tilde{\Omega}$ such that $E\left[\int_0^\infty \int_{\mathbf{R}} 1_{A_n} d\nu\right] < \infty$ for all n.

THEOREM 1.4. *Let ν be an optional $\tilde{\mathcal{P}}$-σ-finite random measure. Then there is a unique (up to a P-null set) random measure, ν^p, called the compensator of ν which is predictable and satisfies either of the following equivalent conditions:*

(i) $E\left[\int_0^\infty \int_{\mathbf{R}} W d\nu\right] = E\left[\int_0^\infty \int_{\mathbf{R}} W d\nu^p\right]$ *for any nonnegative $\tilde{\mathcal{P}}$-measurable W;*

(ii) for any $\tilde{\mathcal{P}}$-measurable W such that $\int_0^\cdot \int_{\mathbf{R}} |W| d\nu$ is (locally) integrable,

$\int_0^t \int_{\mathbf{R}} W d\nu - \int_0^t \int_{\mathbf{R}} W d\nu^p$ *is a (local) martingale.*

Proof: See Theorem II.1.8 of Jacod and Shiryaev [17].

Definition 1.5. ν is an *integer-valued random measure* if ν is a random measure and

(i) $\nu(\omega;\{t\}\times\mathbf{R}) \leq 1$ for all ω, t,

(ii) for any $A \in \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{B}(\mathbf{R})$, $\nu(\cdot, A)$ takes values in $\mathbf{N} \cup \{\infty\}$, and

(iii) ν is optional and $\widetilde{\mathcal{P}}$-σ-finite.

In the sequel, we shall assume that any random measure considered is an integer-valued random measure.

PROPOSITION 1.6. *If ν is an (integer-valued) random measure, then*

$$\nu(\omega; dt, dy) = \sum_{(n)} \varepsilon_{(T_n(\omega), \beta_{T_n}(\omega))}(dt, dy),$$

where ε_a is Dirac measure at a, (T_n) is a sequence of stopping times such that $T_n(\omega) \neq T_m(\omega)$ for $n \neq m$ a.s., and (β_t) is an optional process. Thus, for example, $\int_0^t \int_{\mathbf{R}} W(\omega, s, y)\nu(\omega; ds, dy) = \sum_{T_n \leq t} W(T_n, \beta_{T_n})$.

Proof: See Proposition II.1.14 of Jacod and Shiryayev [17].

THEOREM 1.7. *("independent increments"): The following are equivalent:*

(a) ν has a deterministic compensator, say $\nu^p(\omega; \cdot) = m(\cdot)$ for some σ-finite measure m on $(\mathbf{R}_+ \times \mathbf{R}, \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{B}(\mathbf{R}))$ satisfying on $m(\{t\} \times \mathbf{R}) \leq 1$ for all t;

(b) for any $t \in \mathbf{R}_+$ and any $A \in \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{B}(\mathbf{R})$ such that $A \subseteq (s, \infty) \times E$ and $m(A) < \infty$, the random variable $\nu(\cdot, A)$ is independent of $\mathcal{F}_s$.

Proof: See Theorem II.4.8 and Definition II.1.20 of the "extended Poisson random measure" in [17].

PROPOSITION 1.8. *Let ν be an (integer-valued) random measure. If there exists a version of its compensator ν^p that satisfies identically $\nu^p(w; \{t\} \times \mathbf{R}) = 0$, then the jump times (T_n) of ν are totally inaccessible.*

Proof: See Proposition I.2.26 and Corollary II.1.19 of [17].

Throughout the remainder of this paper, we shall only consider (integer-valued) random measure, ν, with deterministic compensators, ν^p, that satisfy $\nu^p(\omega, \{t\}, \mathbf{R}) = 0$. One implication of Proposition 1.8 for the contingent claims analysis of Chapter 4 is that the stock price there is assumed to jump only at totally inaccessible jump times. This seems reasonable, for if a jump time were predictable, there would be an arbitrage opportunity.

2. Dynamics

We assume throughout that a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ is given and that the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. Consider the stochastic differential equation

$$(2.1) \qquad dx_t = \mu(t, x_{t-})dt + \int_{\mathbf{R}} \gamma(t, x_{t-}, y)(\nu(dt, dy) - \nu^p(dt, dy))$$

for $0 \leq t \leq T$, where we make the following assumptions:

(a) $\nu = \nu(w; dt, dy)$ is an integer-valued random measure on $[0, T] \times \mathbf{R}$ with deterministic compensator $\nu^p(dt, dy) = dtH(dy)$.

(b) $\mu : [0, T] \times \mathbf{R} \to \mathbf{R}$ is a Borel function, continuous in t and three times continuously differentiable in x with bounded derivatives of each order.

(c) $\int_{\mathbf{R}} \gamma(t, x, y)H(dy)$ is continuous in t, differentiable in x with bounded derivative, and is of linear growth.

Note that (a) is quite general by [6] (see also Remark 2.28, Section III.2c of [16]). If X is a diffusion with jumps having third characteristic of the form

$$\nu(w; dt, dy) = dt \times H_t(X_t(w), dy),$$

then under weak assumptions there is a Borel function $\gamma : [0,T] \times \mathbf{R} \times \mathbf{R}$ and a σ-finite measure H such that

$$H_t(x,A) = \int_{\mathbf{R}\setminus\{0\}} 1_A(\gamma(t,x,y))H(dy) \tag{2.2}$$

provided H is infinite and has no atom. In other words, if the compensator ν^p is random, yet Markov, equation (2.2) allows us to imply a deterministic compensator as in equation (2.1).

For $s \leq t$, we write $\xi_{s,t}(x)$ for the unique (strong) solution of (2.1) having initial condition $\xi_{s,s}(x) = x$, that is

$$\xi_{s,t}(x) = x + \int_s^t \mu(r, \xi_{s,r-}(x))dr + \int_s^t \int_{\mathbf{R}} \gamma(r, \xi_{s,r-}(x), y)(\nu(dr,dy) - H(dy)dr).$$

The following lemmas are well known, and we omit the proofs.

LEMMA 2.1. *The solution process, ξ, satisfies the semigroup property:* $\xi_{s,t}(\xi_{0,s}(x_0))$ $= \xi_{0,t}(x_0)$.

Proof: This follows from the uniqueness of the strong solution of the stochastic differential equation.

LEMMA 2.2. *The solution process, $\xi_{s,t}(x)$, is continuously differentiable in x.*

Proof: (See [22], Theorem 36.9, p. 258.)

LEMMA 2.3. *The process ξ is Markov with transition probability $P(s,t,x,A) =$ $P(\xi_{s,t}(x) \in A)$ for $0 \leq s \leq t \leq T$, $x \in \mathbf{R}$, $A \in \mathcal{B}(\mathbf{R})$.*

Proof: (See, e.g., [14], p. 288.)

Remarks 2.4: When there is no possibility of confusion, we shall write ξ_t for $\xi_{0,t}(x_0)$. Suppose $c : \mathbf{R} \to \mathbf{R}$ is twice continuously differentiable and that $E|c(\xi_T)|^2 < \infty$. By Lemma 2.3, we have for $x \in \mathbf{R}$,

$$V_0(t,x) := E[c(\xi_T) \mid \xi_t = x] = E[c(\xi_{t,T}(x))],$$

so that $V_0(t,\xi_t) = E[c(\xi_T) \mid \mathcal{F}_t]$. It follows from Lemmas 2.2 and 2.3 that $V_0(t,x)$ is continuously differentiable in x; in fact

$$\frac{\partial V_0}{\partial x}(t,x) = E\Big[c_\xi(\xi_{t,T}(x))\frac{\partial \xi_{t,T}}{\partial x}(x)\Big].$$

We now use the differentiability in x to prove that $\frac{\partial V_0}{\partial t}$ exists. But first, we need the following simple result.

CLAIM 2.5. *If $\phi : [a,b] \to \mathbf{R}$ is right continuous with finite left limits, then*

(i) for any $x \in (a,b]$,

$$\lim_{h \to 0+} \frac{1}{h} \int_{x-h}^{x} \phi(t)dt = \phi(x-), \quad \text{and}$$

(ii) for any $x \in [a,b)$,

$$\lim_{h \to 0+} \frac{1}{h} \int_{x}^{x+h} \phi(t)dt = \phi(x+) = \phi(x).$$

Proof: (i) Without loss of generality, let $a = 0$ and $b = x = 1$. Fix $\varepsilon > 0$. As ϕ has finite left limit at 1, find $h_0 > 0$ such that

$$|\phi(1-) - \phi(x)| < \varepsilon \qquad \text{for any } 0 < 1 - x < h_0.$$

Then for $h < h_0$,

$$\Big|\frac{1}{h}\int_{1-h}^{1} \phi(x)dx - \phi(1-)\Big| = \Big|\frac{1}{h}\int_{1-h}^{1} [\phi(x) - \phi(1-)]dx\Big|$$

$$\leq \frac{1}{h}\int_{1-h}^{1} |\phi(x) - \phi(1-)|dx < \frac{1}{h}\int_{1-h}^{1} \varepsilon dx = \varepsilon.$$

The proof of (ii) is similar.

LEMMA 2.6. *The derivative* $\frac{\partial V_0(t,x)}{\partial t}$ *exists and is given by*

$$\frac{\partial V_0(t,x)}{\partial t} = -\frac{\partial V_0}{\partial x}(t,x)\mu(t,x) - \int_{\mathbf{R}}[V_0(t,x+\gamma(t,x,y)) - V_0(t,x)]H(dy)$$
$$+ \int_{\mathbf{R}} \frac{\partial V_0(t,x)}{\partial x}\gamma(t,x,y)H(dy).$$

Proof: We first find the left-hand derivative, and so for $h > 0$, consider $V_0(t-h,x) - V_0(t,x)$. Note that

$$V_0(t,\xi_{t-h,t}(x)) = E[c(\xi_T) \mid \xi_t = \xi_{t-h,t}(x)]$$
$$= E[c(\xi_{t,T}(\xi_{t-h,t}(x))) \mid \xi_t = \xi_{t-h,t}(x)]$$

so that

$$E[V_0(t,\xi_{t-h,t}(x))] = E[c(\xi_{t,T}(\xi_{t-h,t}(x)))] = E[c(\xi_{t-h,T}(x))] = V_0(t-h,x).$$

Now, for fixed t, and for $t-h \le s \le t$, we have

$$V_0(t,\xi_{t-h,s}(x)) = V_0(t,x) + \int_{t-h}^{s} \frac{\partial V_0}{\partial x}(t,\xi_{t-h,r-}(x))d\xi_{t-h,r}(x)$$
$$+ \sum_{t-h<r\le s} V_0(t,\xi_{t-h,r}(x)) - V_0(t,\xi_{t-h,r-}(x))$$
$$- \frac{\partial V_0}{\partial x}(t,\xi_{t-h,r-}(x))\Delta\xi_{t-h,r}(x)$$
$$= V_0(t,x) + \int_{t-h}^{s} \frac{\partial V_0}{\partial x}(t,\xi_{t-h,r-}(x))\mu(r,\xi_{t-h,r-}(x))dr$$
$$+ \int_{t-h}^{s}\int_{\mathbf{R}} \frac{\partial V_0}{\partial x}\gamma(r,\xi_{t-h,r-}(x),y)\tilde{\nu}(dr,dy)$$
$$+ \int_{t-h}^{s}\int_{\mathbf{R}} [V_0(t,\xi_{t-h,r-}(x) + \gamma(r,\xi_{t-h,r-}(x),y))$$
$$- V_0(t,\xi_{t-h,r-}(x))]\nu(dr,dy)$$
$$- \int_{t-h}^{s}\int_{\mathbf{R}} \frac{\partial V_0}{\partial x}(t,\xi_{t-h,r-}(x))\gamma(r,\xi_{t-h,r-}(x),y)\nu(dr,dy).$$

Setting $s = t$ and taking expectations, we find that

$$V_0(t-h,x) - V_0(t,x) = \int_{t-h}^{t} E\Big[\frac{\partial V_0}{\partial x}\mu(r,\xi_{t-h,r-}(x))\Big]dr$$

$$+E\Big[\int_{t-h}^{t}\int_{\mathbf{R}}[V_0(t,\xi_{t-h,r-}(x)+\gamma(r,\xi_{t-h,r-}(x),y))-V_0(t,\xi_{t-h,r-}(x))]dr\,H(dy)\Big],$$

because $E\Big[\int_{t-h}^{t}\int_{\mathbf{R}}(\cdot)d\tilde{\nu}\Big] = 0$. Thus $V_0(t-,x) = V_0(t,x)$, and the left-hand derivative is

$$-\lim_{h\to 0+}\frac{V_0(t-h,x)-V_0(t,x)}{h} = -\frac{\partial V_0(t,x)}{\partial x}\mu(t,x)$$

$$-\int_{\mathbf{R}}[V_0(t,x+\gamma(t,x,y))-V_0(t,x)]H(dy) + \int_{\mathbf{R}}\frac{\partial V_0}{\partial x}(t,x)\gamma(t,x,y)H(dy).$$

Here we have used the continuity of μ and γ in the variable t, and the above claim.

The proof for the right-hand derivative is similar: we simply consider, for $h > 0$, $V_0(t+h,x) - V_0(t,x)$, where $V_0(t,x) = E[V_0(t+h,\xi_{t,t+h}(x))]$, and apply the Itô rule to $V_0(t+h,\xi_{t,t+h}(x))$, (fixing the first coordinate).

3. Change of Measure

In various applications, (e.g. contingent claims analysis in finance), one is interested in some alternative probability measure that is equivalent to the original measure P. We shall look at equivalent measures that preserve the Markov properties of our process ξ.

Consider a Girsanov density M defined by the P-martingale

(3.1)

$$M_{0,t}(x_0) = 1 + \int_0^t \int_{\mathbf{R}} M_{0,s-}(x_0)[h_s(\xi_{0,s-}(x_0), y) - 1](\nu(ds, dy) - H(dy)ds)$$

$$= \exp\left\{ \int_0^t \int_{\mathbf{R}} [1 - h_s(\xi_{0,s-}(x_0), y)]dsH(dy) \right\}$$

$$\times \prod_n \{1 + (h_{T_n}(\xi_{T_n^-}, y_n) - 1)1_{\{\Delta\xi_{T_n} \neq 0\}}\}$$

where $T_n(\omega)$ and $y_n(\omega)$ are the jump times and jump sizes, respectively, corresponding to the random measure ν. (Recall that $\xi_s = \xi_{0,s}(x_0)$.) Assuming that $E[M_{0,T}(x_0)] = 1$, and that $h > 0$, define the equivalent probability measure P_h by

(3.2)
$$\frac{dP^h}{dP}\Big|_{\mathcal{F}_t} = M_{0,t}(x_0).$$

Note that we shall also write

(3.3)
$$M_{t,T}(x) = 1 + \int_t^T \int_{\mathbf{R}} M_{t,s-}(x)[h_s(\xi_{t,s-}(x), y) - 1]d\tilde{\nu},$$

so that

(3.4)
$$M_{0,T}(x_0) = M_{0,t}(x_0)M_{t,T}(\xi_t).$$

Again by [22], under suitable conditions for h, there is a version of the solution of (3.3) that is continuously differentiable in x. It is well known (see e.g. [17] Section III.3c) that under P_h, the random measure ν has compensator

(3.5)
$$\nu_h^p(dt, dy) = h_t(\xi_{0,t-}(x_0), y)dtH(dy).$$

It follows that ξ can be written

(3.6)

$$\xi_{0,T}(x_0) = x_0$$

$$+ \int_0^t \Big[\mu(s, \xi_{0,s-}(x_0) + \int_{\mathbf{R}} (h_s(\xi_{0,s-}(x_0), y) - 1)\gamma(s, \xi_{0,s-}(x_0), y)H(dy)\Big] ds$$

$$+ \int_0^t \int_{\mathbf{R}} \gamma(x, \xi_{0,s-}(x_0), y)(\nu(ds, dy) - h ds H(dy)),$$

where the last term is a martingale under P^h.

Now, letting E^h denote expectation under P^h, define

$$V(t, x) = E^h[c(\xi_T) \mid \xi_t = x]. \tag{3.7}$$

Note that $V(t, \xi_t)$ is a P^h-martingale.

LEMMA 3.1. $V(t, x) = E[M_{t,T}(x)c(\xi_{t,T}(x))]$.

Proof: We have

$$V(t, \xi_t) = E^h[c(\xi_{0,T}(x_0)) \mid \xi_t] = E^h[c(\xi_{0,T}(x_0)) \mid \mathcal{F}_t]$$

$$= \frac{E[M_{0,T}(x_0)c(\xi_{0,T}(x_0)) \mid \mathcal{F}_t]}{E[M_{0,T}(x_0) \mid \mathcal{F}_t]} = E[M_{t,T}(\xi_t)c(\xi_{0,T}(x_0)) \mid \mathcal{F}_t]$$

$$= E[M_{t,T}(\xi_t)c(\xi_{t,T}(\xi_t)) \mid \xi_t],$$

where the third equality follows from Bayes' rule, and the fourth, from (2.4). The result follows from Lemma 3.

Remark 3.2: Note that by a similar argument to Lemma 2.4 we can show that $\frac{\partial V}{\partial t}$ exists and is given by

$$\frac{\partial V}{\partial s} = -\frac{\partial V}{\partial x}\mu(s, \xi_{s-}) + \frac{\partial V}{\partial x}\int_{\mathbf{R}} \gamma(s, \xi_{s-}, y)H(dy)$$

$$- \int_{\mathbf{R}} [V(s, \xi_{s-} + \gamma_s(\xi_{s-}, y)) - V(s, \xi_{s-})]h_s(\xi_{s-}, y)H(dy). \tag{3.8}$$

THEOREM 3.3. *Under P^h, the martingale $V(t,\xi_t)$ has the representation*

$$V(t,\xi_t) = E^h[c(\xi_T)] + \int_0^t \int_{\mathbf{R}} \phi_s(y)[\nu(ds,dy) - h_s(\xi_{s-},y)dsH(dy)], \tag{3.9}$$

where

$$\begin{aligned}\phi_s(y) &= E\Big[M_{s,T}(\xi_{s-})\Big\{c(\xi_{s,T}(\xi_{s-}+\gamma_s(\xi_{s-},y)))\\ &\quad\times[L_{s,T}(\xi_{s-},\xi_{s-}+\gamma_s(\xi_{s-},y))+1]-c(\xi_{s,T}(\xi_{s-}))\Big\}\Big]\\ &= E^h\Big[c(\xi_{s,T}(\xi_{s-}+\gamma_s(\xi_{s-},y)))\\ &\quad\times[L_{s,T}(\xi_{s-},\xi_{s-}+\gamma_s(\xi_{s-},y))+1]-c(\xi_{s,T}(\xi_{s-}))\mid\mathcal{F}_{s-}\Big]\end{aligned} \tag{3.10}$$

and, putting $x=\xi_{s-}$ and $\gamma(y)=\gamma_s(\xi_{s-},y)$,

$$\begin{aligned}L_{s,t}(\xi_{s-},\xi_{s-}+\gamma_s(\xi_{s-},y)) &= L_{s,t}(x,x+\gamma(y))\\ &= \int_s^t\int_{\mathbf{R}} \frac{h_r(\xi_{s,r-}(x+\gamma(y)),z)-h_r(\xi_{s,r-}(x),z)}{h_r(\xi_{s,r-}(x),z)}\\ &\qquad\cdot\frac{M_{s,r-}(x+\gamma(y))}{M_{s,r-}(x)}[\nu(dr,dz)-h_r(\xi_{s,r-}(x),z)drH(dz)].\end{aligned} \tag{3.11}$$

Proof. By the Itô rule,

$$\begin{aligned}V(t,\xi_{0,t}(x_0)) &= V(0,x_0)+\int_0^t\frac{\partial V}{\partial s}(s,\xi_{s-})ds+\int_0^t\frac{\partial V}{\partial x}(s,\xi_{s-})d\xi_s\\ &\quad+\int_0^t\int_{\mathbf{R}}\Big[V(s,\xi_s+\gamma_s(\xi_{s-},y))-V(s,\xi_{s-})\\ &\quad-\frac{\partial V}{\partial x}\cdot\gamma(s,\xi_{s-},y)\Big]\nu(ds,dy)\\ &= V(0,x_0)+\int_0^t\int_{\mathbf{R}}[V(s,\xi_{s-}+\gamma_s(\xi_{s-},y))-V(s,\xi_{s-})]\\ &\quad\times(\nu(ds,dy)-h_s(\xi_{s-},y)dsH(dy))\end{aligned} \tag{3.12}$$

after simplifying, and by Remark 3.2.

Thus, if we again put $x = \xi_{s-}$ and $\gamma(y) = \gamma_s(\xi_{s-}, y)$, then the integrand in (3.12) is of the form

$$\begin{aligned}
&V(s, x+\gamma(y)) - V(s,x) \\
&\quad = E[M_{s,T}(x+\gamma(y))c(\xi_{s,T}(x+\gamma(y))) - M_{s,T}(x)c(\xi_{s,T}(x))] \\
&\quad = E\{[M_{s,T}(x+\gamma(y)) - M_{s,T}(x)]c(\xi_{s,T}(x+\gamma(y))) \\
&\qquad + M_{s,T}(x)[c(\xi_{s,T}(x+\gamma(y))) - c(\xi_{s,T}(x))]\}.
\end{aligned}$$

Therefore, it suffices to show that

$$M_{s,T}(x+\gamma(y)) - M_{s,T}(x) = M_{s,T}(x)L_{s,T}(x, x+\gamma(y)).$$

By the product rule, for $s \le t \le T$,

$$\begin{aligned}
&M_{s,t}(x)L_{s,t}(x, x+\gamma(y)) \\
&\quad = \int_s^t M_{s,r-}(x)dL_{s,r} + \int_s^t L_{s,r-}(x, x+\gamma(y))dM_{s,r} \\
&\qquad + \sum_{s<r\le t} \Delta M_{s,r}(x)\Delta L_{s,r}(x, x+\gamma(y)) \\
&\quad = \int_s^t \int_{\mathbf{R}} M_{s,r-}(x)\Big[\frac{h_r(\xi_{s,r-}(x+\gamma(y)), z) - h_r(\xi_{s,r-}(x), z)}{h_r(\xi_{s,r-}(x), z)}\Big] \\
&\qquad \times \frac{M_{s,r-}(x+\gamma(y))}{M_{s,r-}(x)}(\nu(dr,dz) - h_r(\xi_{s,r-}(x), z)\nu^p(dr,dz)) \\
&\qquad + \int_s^t \int_{\mathbf{R}} L_{s,r-}(x, x+\gamma(y))M_{s,r-}(x) \\
&\qquad \times [h_r(\xi_{s,r-}(x), z) - 1](\nu(dr,dz) - \nu^p(dr,dz))
\end{aligned}$$

$$+\int_s^t\int_{\mathbf{R}} M_{s,r-}(x)[h_r(\xi_{s,r-}(x),z)-1]$$
$$\times\Big[\frac{h_r(\xi_{s,r-}(x+\gamma(y)),z)-h_r(\xi_{s,r-}(x),z)}{h_r(\xi_{s,r-}(x),z)}\Big]$$
$$\times\frac{M_{s,r-}(x+\gamma(y))}{M_{s,r-}(x)}\nu(dr,dz)$$
$$=\int_s^t\int_{\mathbf{R}}\Big\{M_{s,r-}(x+\gamma(y))[h_r(\xi_{s,r-}(x+\gamma(y)),z)-h_r(\xi_{s,r-}(x),z)]$$
$$+L_{s,r-}(x,x+\gamma(y))M_{s,r-}(x)[h_r(\xi_{s,r-}(x),z)-1]\Big\}\tilde{\nu}(dr,dz).$$

On the other hand

$$M_{s,t}(x+\gamma(y))-M_{s,t}(x)$$
$$=\int_s^t\int_{\mathbf{R}}\Big\{M_{s,r-}(x+\gamma(y))[h_r(\xi_{s,r-}(x+\gamma(y)),z)-1]$$
$$-M_{s,r-}(x)[h_r(\xi_{s,r-}(x),z)-1]\Big\}\tilde{\nu}(dr,dz)$$
$$=\int_s^t\int_{\mathbf{R}}\Big\{M_{s,r-}(x+\gamma(y))[h_r(\xi_{s,r-}(x+\gamma(y)),z)-h_r(\xi_{s,r-}(x),z)]$$
$$+(M_{s,r-}(x+\gamma(y))-M_{s,r-}(x))[h_r(\xi_{s,r-}(x),z)-1]\Big\}\tilde{\nu}(dr,dz).$$

Thus $M_{s,t}(x)L_{s,t}(x,x+\gamma(y))$ and $M_{s,t}(x+\gamma(y))-M_{s,t}(x)$ satisfy the same stochastic differential equation, and so must be equal.

Example: Suppose N is a nondecreasing process such that $\Delta N_t = 0$ or 1 and

$$Q_t := N_t - \int_0^t \lambda(s,N_{s-})ds \tag{3.13}$$

is a martingale, where λ is bounded, say $0 < \lambda(s,x) \le L$, continuous in s and continuously differentiable in x. Our goals are first to show that N is of type (2.1), and second, to find the martingale representation under some change of measure.

Consider the process

$$(3.14)\quad N_{0,t}(x_0) = x_0 + \int_0^t \mu_s(N_{s-})ds + \int_0^t \int_{\mathbf{R}} \gamma_s(N_{s-}, y)(\nu(ds, dy) - H(dy)ds).$$

We want to find μ and γ such that N_t and $N_{0,t}(x_0)$ have the same law. Since λ is bounded, we may assume that H is finite. For convenience, put $H(dy) = 1_{[A,\infty)}\frac{dy}{y^2}$, where $A = \frac{1}{L} > 0$. Because $\Delta N = 0$ or 1, γ must be an indicator function. In addition, by the martingale property of Q and the uniqueness of compensators, we must have

$$\mu_s(N_{s-}) = \int_{\mathbf{R}} \gamma_s(N_{s-}, y) 1_{[A,\infty)} \frac{dy}{y^2} ds = \lambda_s(N_{s-}).$$

So, for any $s \in [0,T]$, $x \in \mathbf{R}$, let $\alpha(s,x) > 0$ be the value such that

$$\int_{\alpha(s,x)}^{\infty} 1_{[A,\infty)} \frac{dy}{y^2} = \lambda(s,x),$$

i.e., $\frac{1}{\alpha(s,x)} = \lambda(s,x)$, as long as $\alpha \geq A = \frac{1}{L}$ (or equivalently $\lambda \leq L$). It follows that $\gamma_s(x,y) = 1_{[\alpha(s,x),\infty)}(y)$.

Now, γ is not differentiable in x, but we can nevertheless show that $\frac{dN_{s,t}(x)}{dx}$ exists, where the limit is taken in the almost sure sense. By Proposition 1.6, we can write ν in the form

$$\nu(\omega; dt, dy) = \sum_{(n)} \varepsilon_{(T_n(\omega), y_n(\omega))}(dt, dy).$$

Clearly, if $E = \{\omega : y_n(\omega) = \alpha(T_n(\omega), N_{0,T_n^-}(x_0)(\omega)) \text{ for any } n\}$, then $P(E) = 0$. Thus, for a fixed $\omega \notin E$, there exists an $\varepsilon = \varepsilon(\omega) > 0$ such that

$$y_n(\omega) \in [\alpha(T_n(\omega), N_{0,T_n^-}(x_0)(\omega) \pm \varepsilon(\omega)), \infty) \quad \text{iff}$$

$$y_n(\omega) \in [\alpha(T_n(\omega), N_{0,T_n^-}(x_0)(\omega)), \infty)$$

for all n such that $T_n(\omega) \leq T$, since there are only finitely many jumps. Now let $\delta(\omega)$ be such that $|\alpha(t, x+\Delta x) - \alpha(t,x)| < \varepsilon$ for all $-\delta < \Delta x < \delta$. It follows that

$$y_n(\omega) \in [\alpha(T_n, N_{0,T_n^-}(x_0+\Delta x)), \infty) \quad \text{iff}$$

$$y_n(\omega) \in [\alpha(T_n, N_{0,T_n^-}(x_0)), \infty)$$

for all n such that $T_n(\omega) \leq T$, that is, $N_{0,t}(x_0+\Delta x) = x_0 + \Delta x$ for $t < T_1(\omega)$ and $N(x_0+\Delta x)$ does not pick up any more (or less) jumps than $N(x_0)$. Thus $N_t(x_0+\Delta x)(\omega) - N_t(x_0)(\omega) = \Delta x$, for $-\delta < \Delta x < \delta$, so that $\frac{dN_t(x_0)}{dx} = 1$. Therefore, we can apply Theorem 3.3.

Consider a Girsanov density M defined by

$$M_{0,t}(x_0) = 1 + \int_0^t M_{0,s-}(x_0)(h_s(N_{0,s-}(x_0)) - 1)dQ_s.$$

As before, define the equivalent measure P^h by setting $\frac{dP^h}{dP}\Big|_{\mathcal{F}_t} = M_{0,t}(x_0)$. Analogously to our previous notation, write $N_{s,t}(x)$ for the process starting at x at time s, with obvious interpretations for $Q_{s,t}(x)$ and $M_{s,t}(x)$.

Now, let $V(t,N_t) = E^h[c(N_T) \mid \mathcal{F}_t] = E[M_{t,T}(x)c(N_{t,T}(x)) \mid x]$ where $x = N_t$. By Theorem 3.3,

$$V(t,N_t) = V(0,x_0) + \int_0^t [V(s,N_{s-}+1) - V(s,N_{s-})](dN_s - h_s(N_{s-})\lambda(N_{s-})ds),$$

where the integrand can be written

$$V(s,N_{s-}+1) - V(s,N_{s-}) = V(s,x+1) - V(s,x)$$

$$= E^h[c(N_{s,T}(x+1))(L_{s,T}(x,x+1)+1) - c(N_{s,T}(x)) \mid x]$$

where

$$L_{s,T}(x,x+1) = \int_s^T \left[\frac{h_r(N_{s,r-}(x+1)) - h_r(N_{s,r-}(x))}{h_r(N_{s,r-}(x))}\right] \frac{M_{s,r-}(x+1)}{M_{s,r-}(x)}$$

$$\cdot (dN_r - h_r(N_{r-})\lambda_r(N_{r-})dr).$$

Chapter 3

DIFFUSIONS WITH JUMPS

1. Dynamics

As in the pure jump case, we assume that a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ is given, and that the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. We now consider the following stochastic differential equation defined for $0 \le t \le T$,

$$\begin{aligned} dx_t &= \mu(t, x_t)dt + \sigma(t, x_t)dW_t \\ &\quad + \int_{\mathbf{R}} \gamma(t, x_{t-}, y)(\nu(dt, dy) - H(dy)dt) \end{aligned} \tag{1.1}$$

where W is standard Brownian motion, ν is an integer-valued random measure with deterministic compensator $H(dy)dt$, and W and ν are independent. We shall give the hypotheses for the coefficients μ, σ, and γ below. For now, let us write $\xi_{s,t}(x)$ $(s \le t)$ for the unique strong solution (assumed to exist) of (1.1) having initial condition $\xi_{s,s}(x) = x$.

We shall now summarize the properties of ξ that we will be using. The proofs for the following lemmas were briefly indicated in the pure jump case, and little is changed in the case of diffusions with jumps.

THEOREM 1.1. *Assume the following:*

(A1) *for* $k = 1, 2$, $\dfrac{\partial^k \mu_t(x)}{\partial x}$, *and* $\dfrac{\partial^k \sigma_t(x)}{\partial x}$, *exist, are continuous and bounded;*

(A2) *for* $k = 2, 3, 4$,

$$\int_{\mathbf{R}} \left\{ |\gamma_t(x, y)|^k + \left| \frac{\partial \gamma_t}{\partial x}(x, y) \right|^k \right\} H(dy) \le C;$$

(A3)
$$\int \left| \frac{\partial \gamma_t}{\partial x}(x, y) - \frac{\partial \gamma_{t'}}{\partial x}(x', y) \right| H(dy) \to 0$$

as $(t', x') \to (t, x)$. *Then (i) there exists exists a unique solution* $\xi_{s,t}(x)$ *to equation (1.1) such that* ξ *is square integrable; (ii)* $\frac{\partial \xi}{\partial x}$ *exists, both when the derivative is understood as the limit of the corresponding quotient in the mean square sense and in the almost sure sense; (iii)* $\frac{\partial^2 \xi}{\partial x^2}$ *exists in the mean square sense; (iv) the martingale part of* ξ *is a square integrable martingale.*

Proof: See [14], Theorem 1, p. 275 for the existence and uniqueness of ξ, together with the square integrability of ξ. Métivier ([22], Theorem 36.9) proves the existence of $\frac{\partial \xi_\tau}{\partial x}$ in the almost sure sense. For the mean-square derivatives, see [14]; Theorem 5, p. 280, and the remarks following establish the existence of the first derivative, and Theorem 7, p. 287, proves the existence of the second derivative.

The proof of Theorem 1, p. 275 of [14] can also be used to show that there exists a constant L such that

$$E[\xi_\tau^2(x_0)] \leq L(1 + |x_0|^2)$$

for any stopping time τ, $0 \leq \tau \leq T$. Clearly, then, the martingale part of ξ, M, also satisfies $\sup_\tau E[M_\tau^2] < \infty$. Thus the set of random variables $\{M_\tau\}$ is uniformly integrable (by, eg. Corollary 1.19 of [7]), that is, M is of class D. Therefore, M is a uniformly integrable martingale (Proposition 1.47(c) of [17]).

LEMMA 1.2. *Under the hypotheses of Theorem1.1,* ξ *satisfies the semigroup property*

$$\xi_{s,t}(\xi_{r,s}(x)) = \xi_{r,t}(x) \quad \text{for } 0 \leq r \leq s \leq t \leq T.$$

Proof: This follows from the uniqueness of the solutions of the stochastic differential equations (1.1), which is implied by the hypotheses of Theorem 1.1.

LEMMA 1.3. *The process ξ is Markov with transition probability $P(s,t,x,A) = P(\xi_{s,t}(x) \in A)$, $0 \le s \le t \le T$, $x \in \mathbf{R}$, $A \in \mathcal{B}(\mathbf{R})$.*

Proof: See, e.g., [14], p. 288.

For various applications, we would like to consider martingales of the form $E[c(\xi_T) \mid \mathcal{F}_t] = E[c(\xi_T) \mid \xi_t]$, where $c : \mathbf{R} \to \mathbf{R}$ is assumed to be twice differentiable with bounded derivatives. (Note: we always take right continuous versions of such martingales.) This martingale is associated with the (Borel) measurable function

$$V_0(t,x) = E[c(\xi_T) \mid \xi_t = x] = E[c(\xi_{t,T}(x))]$$

by the relation $V_0(t,\xi_t) = E[c(\xi_T) \mid \mathcal{F}_t]$. Note that $V_0(t,x)$ is clearly twice differentiable in x; in fact $\frac{\partial V_0}{\partial x} = E\left[c_\xi(\xi_{t,T}(x)) \cdot \frac{\partial \xi_{t,T}}{\partial x}(x)\right]$. The following lemma establishes the differentiability in t.

LEMMA 1.4. *The derivative $\frac{\partial V_0}{\partial t}$ exists and satisfies the integro-differential equation*

$$(1.2)\quad \frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial x} \cdot \mu_t + \frac{1}{2}\frac{\partial^2 V_0}{\partial x^2}\sigma_t^2 - \int_{\mathbf{R}} \frac{\partial V_0}{\partial x}\gamma_t(x,y)H(dy) + \int_{\mathbf{R}} [V_0(t,x+\gamma_t(x,y)) - V_0(t,x)]H(dy) = 0$$

with terminal condition $V_0(T,x) = c(x)$.

Proof: The proof follows by analogy to the proof of Lemma 2.6 of Chapter 2.

2. Change of Measure

In the previous section, we reviewed the main properties of interest for the model that we are considering, namely the semigroup and Markov properties for ξ, the differentiability of ξ as a function of its initial value, and the differentiability of $V_0(t,x) = E[c(\xi_T) \mid \xi_t = x]$ in t and in x. In this section, we shall again introduce an equivalent change of probability measure of the Girsanov type. We are particularly interested in measures under which ξ becomes a martingale. We then define the function $V(t,x)$ analogously to the function $V_0(t,x)$ defined above, except that the conditional expectation is taken with respect to the new measure. The process $V(t,\xi_t)$ is again a martingale, but under the new measure. Our goal is to supply a martingale representation result for $V(t,\xi_t)$.

Consider the Girsanov density

(2.1)

$$\begin{aligned} G_{0,t}(x_0) &= 1 + \int_0^t G_{0,s}(x_0) g_s(\xi_{0,s}(x_0)) dW_s \\ &\quad + \int_0^t \int_{\mathbf{R}} G_{0,s-}(x_0)[h_s(\xi_{0,s-}(x_0), y) - 1](\nu(ds,dy) - H(dy)ds) \\ &= \exp\Big\{ \int_0^t g_s(\xi_s) dW_s - \frac{1}{2}\int_0^t g_s^2(\xi_s) ds - \int_0^t \int_{\mathbf{R}} [h_s(\xi_{s-}, y) - 1] H(dy) ds \\ &\quad + \int_0^t \int_{\mathbf{R}} \ln(h_s(\xi_{s-}, y)) \nu(dy, ds) \Big\} \end{aligned}$$

and define $P^{g,h}$ by

$$\frac{dP^{g,h}}{dP}\Big|_{\mathcal{F}_t} = G_{0,t}(x_0).$$

Here we assume that G exists and is a square integrable martingale.

Let $E^{g,h}$ denote expectation with respect to $P^{g,h}$. Then, we write $V(t,x) := E^{g,h}[c(\xi_T) \mid \xi_t = x]$. Recall that

$$\xi_{s,t}(x) = x + \int_s^t \mu_r(\xi_{r-})dr + \int_s^t \sigma_r(\xi_{r-})dW_r + \int_s^t \int_{\mathbf{R}} \gamma_r(\xi_{r-}, y)(\nu(dr,dy) - H(dy)dr)$$

It is well known that under $P^{g,h}$, $W_t - \int_0^t g_r(\xi_{r-})dr$ is a standard Brownian motion, and ν has compensator $h_r(\xi_{r-}, y)H(dy)dr$. So, ξ can be written

(2.2)

$$\begin{aligned}\xi_{s,t}(x) = x &+ \int_s^t \sigma_r(\xi_{r-})(dW_r - g_r(\xi_{r-})dr) \\ &+ \int_s^t \int_{\mathbf{R}} \gamma_r(\xi_{r-}, y)(\nu(dr,dy) - h_r(\xi_{r-}, y)H(dy)dr) \\ &+ \int_s^t \left\{ \mu_r(\xi_{r-}) + \sigma_r(\xi_{r-})g_r(\xi_{r-}) + \int_{\mathbf{R}} \gamma_r(\xi_{r-}, y)[h_r(\xi_{r-}, y) - 1]H(dy) \right\} dr.\end{aligned}$$

Remark 2.1: In financial applications g and h are chosen so that the last term is a.s. zero; often this can be done in infinitely many ways. Then ξ is a martingale under $P^{g,h}$, and in this case, $P^{g,h}$ is called a(n) (equivalent) *martingale measure*. This situation will be discussed in Chapter 4.

Note that we interpret $\xi_{s,t}(x)$ to be the process ξ "starting at position x at time s." On the other hand, let us now define $G_{s,t}(x)$ to be the (unique) strong stolution of the stochastic differential equation

(2.1′)

$$G_{s,t}(x) = 1 + \int_s^t G_{s,r-}(x)g_r(\xi_{s,r}(x))dW_r + \int_s^t \int_{\mathbf{R}} G_{s,r-}(x)[h_r(\xi_{s,r-}(x), y) - 1](\nu(dr,dy) - H(dy)dr).$$

Thus $G_{s,t}(x)$ still "starts at position 1" (as does $G_{0,t}(x_0)$), but it also depends on x, the starting position for ξ. (It may help to think of the pair $(\xi, G) \in \mathbf{R}^2$ as "starting at $(x, 1)$.") In fact G is differentiable with respect to x. This notation allows us to write V in a simple form, in terms of the original probability, as the following lemma shows.

LEMMA 2.2. $V(t,x) = E[G_{t,T}(x)c(\xi_{t,T}(x))]$.

Proof: See Lemma 3.1 of Chapter 2.

We are now ready to present our martingale representation result.

THEOREM 2.3. *Let the processes L and K be defined as follows*

$$L_{s,t} = \int_s^t \frac{\partial g_r}{\partial \xi}(\xi_{r-}) \cdot \frac{\partial \xi_{s,r-}}{\partial x}(x)(dW_r - g_r(\xi_{r-})dr) \tag{2.3}$$

$$+ \int_s^t \int_{\mathbf{R}} h_r(\xi_{r-}, y)^{-1} \cdot \frac{\partial h_r}{\partial \xi}(\xi_{r-}, y) \cdot \frac{\partial \xi_{s,r-}}{\partial x}(x)$$

$$\times (\nu(dr, dy) - h_r(\xi_{r-}, y)H(dy)dr)$$

and

$$K_{s,t} = \int_s^t \frac{G_{s,r-}(\xi_{s-} + \gamma(y))}{G_{s,r-}(\xi_{s-})}[g_r(\xi_{s,r-}(\xi_{s-} + \gamma(y))) - g_r(\xi_{s,r-}(\xi_{s-}))]$$

$$\times (dW_r - g_r(\xi_{s,r-}(\xi_{s-})dr)$$

$$+ \int_s^t \int_{\mathbf{R}} \frac{G_{s,r-}(\xi_{s-} + \gamma(y))}{G_{s,r-}(\xi_{s-})} \tag{2.4}$$

$$\times \left[\frac{h_r(\xi_{s,r-}(\xi_{s-} + \gamma(y)), z) - h_r(\xi_{s,r-}(\xi_{s-}), z)}{h_r(\xi_{s,r-}(\xi_{s-}), z)}\right]$$

$$\times (\nu(dr, dz) - h_r(\xi_{s,r-}(\xi_{s-}), z)H(dz)dr).$$

Here $\gamma(y) = \gamma_s(\xi_{s-}, y)$. *Then the* $P^{g,h}$*-martingale* $V(t, \xi_t)$ *has the representation*

(2.5)

$$V(t,\xi_t) = V(0,x_0) + \int_0^t \phi_r^c(\xi_r)(dW_r - g_r(\xi_r)dr) + \int_0^t \int_{\mathbf{R}} \phi_r^d(\xi_{r-},y)(\nu(dr,dy) - h_r(\xi_{r-},y)H(dy)dr),$$

where

$$\phi_r^c(\xi_r) = E^{g,h}[L_{r,T}c(\xi_{r,T}(x)) + \frac{\partial c}{\partial \xi}(\xi_{r,T}(x))\frac{\partial \xi_{r,T}}{\partial x}(x)\Big] \cdot \sigma_r(\xi_r) \tag{2.6}$$

and

$$\phi_r^d(\xi_{r-},y) = E^{g,h}[(K_{r,T}+1)c(\xi_{r,T}(x_- + \gamma(y))) - c(\xi_{r,T}(x_-))] \tag{2.7}$$

and the above expectations are evaluated at $x = \xi_r$ *and* $x_- = \xi_{r-}$.

The proof is provided by a sequence of lemmas.

LEMMA 2.4.

$$V(t,\xi_t) = V(0,x_0) + \int_0^t \frac{\partial V}{\partial x}(r,\xi_{r-})\sigma_r(\xi_{r-})(dW_r - g_r(\xi_{r-})dr) + \int_0^t \int_{\mathbf{R}} [V(r,\xi_{r-} + \gamma_r(\xi_{r-},y)) - V(r,\xi_{r-})](\nu(dr,dy) - h_r(\xi_{r-},y)H(dy)dr).$$

Proof: By Itô's lemma,

$$V(t,\xi_t) = V(0,x_0) + \int_0^t \frac{\partial V}{\partial r}dr + \int_0^t \frac{\partial V}{\partial x}\mu_r(\xi_{r-})dr + \int_0^t \frac{\partial V}{\partial x}\sigma_r(\xi_{r-})dW_r + \int_0^t \int_{\mathbf{R}} \frac{\partial V}{\partial r}\gamma_r(\xi_{r-},y)(\nu(dr,dy) - H(dy)dr)$$

$$+\frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial x^2}\sigma_r^2(\xi_{r-})dr$$

$$+\int_0^t\int_{\mathbf{R}}[V(r,\xi_{r-}+\gamma_r(\xi_{r-},y)-V(r,\xi_{r-})-\frac{\partial V}{\partial x}\gamma_r(\xi_{r-},y)]\nu(dr,dy)$$

$$=V(0,x_0)+\int_0^t \frac{\partial V}{\partial x}\sigma_r(\xi_r)(dW_r-g_r(\xi_{r-})dr)$$

$$+\int_0^t\int_{\mathbf{R}}[V(r,\xi_{r-}+\gamma_r(\xi_{r-},y)-V(r,\xi_{r-})](\nu(dr,dy)-h_r(\xi_{r-},y)H(dy)dr)$$

$$+\int_0^t\Big\{\frac{\partial V}{\partial r}+\frac{\partial V}{\partial x}\Big[\mu_r(\xi_r)+\sigma_r(\xi_r)g_r(\xi_r)-\int_{\mathbf{R}}\gamma_r(\xi_r,y)H(dy)\Big]+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma_r^2(\xi_r)$$

$$+\int_{\mathbf{R}}[V(r,\xi_{r-}+\gamma_r(\xi_{r-},y))-V(r,\xi_{r-})]h_r(\xi_{r-},y)H(dy)\Big\}dr.$$

Clearly the dr integral is a continuous (local) martingale of finite variation under $P^{g,h}$. Therefore, the dr integral above is a.s. zero, giving us the desired result.

As an immediate consequence of this lemma we have the following corollary which should be compared to equation (1.2).

COROLLARY 2.5. *The function $V(t,x)$ satisfies the integro-differential equation*

$$\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)\Big[\mu_t(x)+\sigma_t(x)g_t(x)-\int_{\mathbf{R}}\gamma_t(x,y)H(dy)\Big]+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma_t^2(x)$$

$$+\int_{\mathbf{R}}[V(t,x+\gamma_t(x,y))-V(t,x)]h_t(x,y)H(dy)=0$$

with terminal condition $V(T,x)=c(x)$.

COROLLARY 2.6. *If $P^{g,h}$ is a martingale measure (cf. Remark 2.1) then V satisfies*

$$\frac{\partial V}{\partial t}(t,x)-\int_{\mathbf{R}}\frac{\partial V}{\partial x}(t,x)\gamma_t(x,y)h_t(x,y)H(dy)+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,x)\sigma_t^2(x)$$

$$+\int_{\mathbf{R}}[V(t,x+\gamma_t(x,y))-V(t,x)]h_t(x,y)H(dy)=0,$$

and $V(T,x) = c(x)$.

Proof: This follows from the proof of the lemma and the fact that $P^{g,h}$ is a martingale measure iff

$$\mu_t(x) + \sigma_t(x)g_t(x) + \int_{\mathbf{R}} \gamma_t(x,y)[h_t(x,y) - 1]H(dy) = 0,$$

as noted in Remark 1.

Proof of the theorem: (1) Consider ϕ_t^c first. By Lemma 2.4, we have $\phi_t^c(\xi_t) = \frac{\partial V}{\partial x}(t,\xi_t) \cdot \sigma_t(\xi_t)$. However, putting $x = \xi_t$,

$$(2.8) \quad \frac{\partial V}{\partial x}(t,x) = E\Big[\frac{\partial G_{t,T}(x)}{\partial x} \cdot c(\xi_{t,T}(x)) + G_{t,T}(x) \cdot \frac{\partial c(\xi_{t,T}(x))}{\partial \xi} \cdot \frac{\partial \xi_{t,T}(x)}{\partial x}\Big],$$

assuming that $\frac{\partial G}{\partial x}$ exists in the mean-square sense. Thus it suffices to show that $\frac{\partial G_{t,T}(x)}{\partial x} = G_{t,T}(x) \cdot L_{t,T}$, for then we would have

$$\begin{aligned}\frac{\partial V}{\partial x}(t,x) &= E\Big[G_{t,T}(x)\Big(L_{t,T}c(\xi_{t,T}(x)) + \frac{\partial c}{\partial \xi}(\xi_{t,T}(x)) \cdot \frac{\partial \xi_{t,T}(x)}{\partial x}\Big)\Big] \\ &= E^{g,h}\Big[L_{t,T}c(\xi_{t,T}(x)) + \frac{\partial c}{\partial \xi}(\xi_{t,T}(x)) \cdot \frac{\partial \xi_{t,T}}{\partial x}(x)\Big].\end{aligned}$$

From equations (2.1′) and (2.3) we recall that $G_{t,T}(x)$ and $L_{t,T}$ are given by

$$\begin{aligned}G_{t,T}(x) = 1 &+ \int_t^T G_{t,r-}(x)g_r(\xi_{t,r}(x))dW_r \\ &+ \int_t^T \int_{\mathbf{R}} G_{t,r-}(x)[h_r(\xi_{t,r-}(x),y) - 1](\nu(dr,dy) - H(dy)dr),\end{aligned}$$

and

$$\begin{aligned}L_{t,T} = &\int_t^T \frac{\partial g_r(\xi_{t,r}(x))}{\partial \xi} \cdot \frac{\partial \xi_{t,r}(x)}{\partial x}(dW_r - g_r(\xi_{t,r}(x))dr) \\ &+ \int_t^T \int_{\mathbf{R}} h_r(\xi_{t,r-}(x),y)^{-1} \cdot \frac{\partial h_r}{\partial \xi}(\xi_{t,r-}(x),y) \\ &\cdot \frac{\partial \xi_{t,r-}(x)}{\partial x}(\nu(dr,dy) - h_r(\xi_{t,r-}(x),y)H(dy)dr).\end{aligned}$$

Now, $\frac{\partial G_{t,T}(x)}{\partial x}$ satisfies the stochastic differential equation

$$\begin{aligned}\frac{\partial G_{t,T}(x)}{\partial x} &= \int_t^T \frac{\partial G_{t,r-}(x)}{\partial x} g_r(\xi_{t,r}(x))dW_r \\ &+ \int_t^T G_{t,r-}(x)\frac{\partial g}{\partial \xi}(\xi_{t,r}(x))\frac{\partial \xi_{t,r}}{\partial x}(x)dW_r \\ &+ \int_t^T \int_{\mathbf{R}} \Big\{\frac{\partial G_{t,r-}(x)}{\partial x}[h_r(\xi_{t,r-}(x),y)-1] \\ &+ G_{t,r-}(x)\frac{\partial h_r}{\partial \xi}(\xi_{t,r-}(x),y)\frac{\partial \xi_{t,r-}}{\partial x}(x)\Big\}(\nu(dr,dy)-H(dy)dr).\end{aligned}$$

On the other hand,

$$\begin{aligned}G_{t,T}(x)L_{t,T} &= \int_t^T G_{t,r-}(x)dh_{t,r} + \int_t^T L_{t,r-}dG_{t,r}(x) \\ &+ \langle G^c, L^c\rangle_{t,T} + \sum_{t<\sigma\le T}\Delta G_{t,r}(x)\Delta L_{t,r}(x) \\ &= \int_t^T G_{t,r-}(x)\frac{\partial g_r}{\partial \xi}(\xi_{t,r-}(x))\frac{\partial \xi_{t,r}}{\partial x}(x)(dW_r - g_r(\xi_{t,r}(x))dr) \\ &+ \int_t^T \int_{\mathbf{R}} G_{t,r-}(x)h_r(\xi_{t,r-}(x),y)^{-1}\frac{\partial h_r}{\partial \xi}(\xi_{t,r-}(x),y) \\ &\times \frac{\partial \xi_{t,r-}}{\partial x}(x)(\nu(dr,dy) - h_r(\xi_{t,r-}(x),y)H(dy)dr) \\ &+ \int_t^T L_{t,r-}G_{t,r-}(x)g_r(\xi_{t,r}(x))dW_r \\ &+ \int_t^T \int_{\mathbf{R}} L_{t,r-}G_{t,r-}(x)[h_r(\xi_{t,r-}(x),y)-1](\nu(dr,dy)-H(dy)dr) \\ &+ \int_t^T G_{t,r-}(x)g_r(\xi_{t,r}(x))\frac{\partial g_r}{\partial \xi}(\xi_{t,r}(x))\frac{\partial \xi_{t,r}}{\partial x}(x)dr \\ &+ \int_t^T \int_{\mathbf{R}} G_{t,r-}(x)[h_r(\xi_{t,r-}(x),y)-1]h_r(\xi_{t,r-}(x),y)^{-1} \\ &\times \frac{\partial h_r}{\partial \xi}(\xi_{t,r-}(x),y)\frac{\partial \xi_{t,r-}}{\partial x}(x)\nu(dr,dy)\end{aligned}$$

$$= \int_t^t L_{t,r-} G_{t,r-}(x) g_r(\xi_{t,r}(x)) dW_r$$

$$+ \int_t^T G_{t,r-}(x) \frac{\partial g_r}{\partial \xi}(\xi_{t,r}(x)) \frac{\partial \xi_{t,r}}{\partial x}(x) dW_r$$

$$+ \int_t^T \int_{\mathbf{R}} \Big\{ G_{t,r-}(x) L_{t,r-}[h_r(\xi_{t,r-}(x), y) - 1]$$

$$+ G_{t,r-}(x) \frac{\partial h_r}{\partial \xi}(\xi_{t,r-}(x), y) \frac{\partial \xi_{t,r-}}{\partial x}(x) \Big\} (\nu(dr, dy) - H(dy)dr).$$

Thus $\frac{\partial G_{t,T}}{\partial x}(x)$ and $G_{t,T}(x)L_{t,T}$ satisfy the same stochastic differential equations, and so by the uniqueness of the solutions, $\frac{\partial G_{t,T}(x)}{\partial x} = G_{t,T}(x)L_{t,T}$, as required.

(2) Now we consider $\phi_t^d(\xi_{t-}, y)$. Again, by Lemma 2.4

$$\phi_t^d(\xi_{t-}, y) = V(t, \xi_{t-} + \gamma_t(\xi_{t-}, y)) - V(t, \xi_{t-}),$$

or, putting $x = \xi_{r-}$ this time, and using Lemma 2.2

$$\phi_t^d(x, y) = V(t, x + \gamma_t(x, y)) - V(t, x)$$

$$= E[G_{t,T}(x + \gamma_t(x, y)) c(\xi_{t,T}(x + \gamma_t(x, y)))]$$

$$- E[G_{t,T}(x) c(\xi_{t,T}(x))]$$

$$= E[(G_{t,T}(x + \gamma_t(x, y)) - G_{t,T}(x)) c(\xi_{t,T}(x + \gamma_t(x, y)))$$

$$+ G_{t,T}(x)(c(\xi_{t,T}(x + \gamma_t(x, y))) - c(\xi_{t,T}(x)))].$$

Therefore, it remains to show that

$$G_{t,T}(x + \gamma_t(x, y)) - G_{t,T}(x) = G_{t,T}(x) \cdot K_{t,T}.$$

First, let us recall that (writing $\gamma_t(\xi_t, y) = \gamma(y)$),

$$K_{t,T} = \int_t^T \frac{G_{t,r-}(\xi_{t-} + \gamma(y))}{G_{t,r-}(\xi_{t-})}[g_r(\xi_{t,r-}(\xi_{t-} + \gamma(y))) - g_r(\xi_{t,r-}(\xi_{t-}))]$$

$$\times (dW_r - g_r(\xi_{t,r-}(\xi_{t-}))dr) + \int_t^T \int_{\mathbf{R}} \frac{G_{t,r-}(\xi_{t-} + \gamma(y))}{G_{t,r-}(\xi_{t-})}$$

$$\times \left[\frac{h_r(\xi_{t,r-}(\xi_{t-} + \gamma(y)), z) - h_r(\xi_{t,r-}(\xi_{t-}), z)}{h_r(\xi_{t,r-}(\xi_{t-}), z)}\right]$$

$$\times (\nu(dr, dz) - h_r(\xi_{t,r-}(\xi_{t-}), z)H(dy)dr).$$

So

$$G_{t,T}(\xi_{t-})K_{t,T} = \int_t^T G_{t,r-}(\xi_{t-})K_{t,r} + \int_t^T K_{t,r-}dG_{t,r}(\xi_{t-})$$

$$+ \langle G^c, K^c\rangle_{t,T} + \sum_{t<r\le T} \Delta G_{t,r}(\xi_{t-})\Delta K_{t,r}$$

$$= \int_t^T G_{t,r-}(\xi_{t-} + \gamma(y))[g_r(\xi_{t,r-}(\xi_{t-} + \gamma(y))) - g_r(\xi_{t,r-}(\xi_{t-}))]$$

$$\times (dW_r - g_r(\xi_{t,r-}(\xi_{t-}))dr) + \int_t^T \int_{\mathbf{R}} G_{t,r-}(\xi_{t-} + \gamma(y))$$

$$\times \left[\frac{h_r(\xi_{t,r-}(\xi_{t-} + \gamma(y)), z) - h_r(\xi_{t,r-}(\xi_{t-}), z)}{h_r(\xi_{t,r-}(\xi_{t-}), z)}\right]$$

$$\times (\nu(dr, dz) - h_r(\xi_{t,r-}(\xi_{t-}), z)H(dz)dr)$$

$$+ \int_t^T [G_{t,r-}(\xi_{t-})K_{t,r-}]g_r(\xi_{t,r-}(\xi_{t-}))dW_r$$

$$+ \int_t^T \int_{\mathbf{R}} [G_{t,r-}(\xi_{t-})K_{t,r}][h_r(\xi_{t,r-}(\xi_{t-}), z) - 1]$$

$$\times (\nu(dr, dz) - H(dz)dr)$$

$$+ \int_t^T G_{t,r-}(\xi_{t-} + \gamma(y))g_r(\xi_{t,r}(\xi_{t-}))$$

$$\times [g_r(\xi_{t,r-}(\xi_{t-}+\gamma(y))) - g_r(\xi_{t,r}(\xi_{t-}))]dr$$

$$+ \int_t^T \int_{\mathbf{R}} G_{t,r-}(\xi_{t-}+\gamma(y)) \Big[\frac{h_r(\xi_{t,r-}(\xi_{t-}+\gamma(y)),z) - h_r(\xi_{t,r-}(\xi_{t-}),z)}{h_r(\xi_{t,r-}(\xi_{t-}),z)} \Big]$$

$$\times [h_r(\xi_{t,r-}(\xi_{t-}),z) - 1]\nu(dr,dz)$$

$$= \int_t^T \Big\{ G_{t,r-}(\xi_{t-}+\gamma(y))[g_r(\xi_{t,r-}(\xi_{t-}+\gamma(y))) - g_r(\xi_{t,r-}(\xi_t))]$$

$$+ [G_{t,r-}(\xi_t)K_{t,r-}]g_r(\xi_{t,r-}(\xi_{t-})) \Big\} dW_r$$

$$+ \int_t^T \int_{\mathbf{R}} \Big\{ G_{t,r-}(\xi_{t-}+\gamma(y))[h_r(\xi_{t,r-}(\xi_{t-}+\gamma(y)),z) - h_r(\xi_{t,r-}(\xi_t),z)]$$

$$+ [G_{t,r-}(\xi_{t-})K_{t,r-}][h_r(\xi_{t,r-}(\xi_{t-}),z) - 1] \Big\} (\nu(dr,dz) - H(dz)dr).$$

On the other hand,

$$G_{t,T}(\xi_{t-}+\gamma(y)) - G_{t,T}(\xi_{t-})$$

$$= \int_t^T \Big\{ G_{t,r-}(\xi_{t-}+\gamma(y))g_r(\xi_{t,r-}(\xi_{t-}+\gamma(y))) - G_{t,r-}(\xi_{t-})g_r(\xi_{t,r-}(\xi_{t-})) \Big\} dW_r$$

$$+ \int_t^T \int_{\mathbf{R}} \Big\{ G_{t,r-}(\xi_{t-}+\gamma(y))[h_r(\xi_{t,r-}(\xi_{t-}+\gamma(y)),z) - 1]$$

$$- G_{t,r-}(\xi_{t-})[h_r(\xi_{t,r-}(\xi_{t-}),z) - 1] \Big\} (\nu(dr,dz) - H(dz)dr)$$

$$= \int_t^T \Big\{ G_{t,r-}(\xi_{t-}+\gamma(y))[g_r(\xi_{t,r-}(\xi_{t-}+\gamma(y))) - g_r(\xi_{t,r-}(\xi_{t-}))]$$

$$+ [G_{t,r-}(\xi_{t-}+\gamma(y)) - G_{t,r-}(\xi_{t-})]g_r(\xi_{t,r-}(\xi_{t-})) \Big\} dW_r$$

$$+ \int_t^T \int_{\mathbf{R}} \Big\{ G_{t,r-}(\xi_{t-}+\gamma(y))[h_r(\xi_{t,r-}(\xi_{t-}+\gamma(y)),z) - h_r(\xi_{t,r-}(\xi_{t-}),z)]$$

$$+ [G_{t,r-}(\xi_{t-}+\gamma(y)) - G_{t,r-}(\xi_{t-})][h_r(\xi_{t,r-}(\xi_t),z) - 1] \Big\} (\nu(dr,dz) - H(dz)dr).$$

Therefore, $G_{t,T}(\xi_{t-})K_{t,T}$ and $G_{t,T}(\xi_{t-} + \gamma_t(\xi_{t-}, y)) - G_{t,T}(\xi_{t-})$ satisfy the same stochastic differential equations and so again by the uniqueness of solutions they must be equal. This completes the proof of the theorem.

Chapter 4

APPLICATION TO CONTINGENT CLAIM ANALYSIS

1. Introduction

In [3] Black and Scholes first described a method of pricing derivative securities depending on non-dividend paying stocks, such as the European option. Their approach was to derive a partial differential equation, the solution of which gave the "value" process of the option. The problem became more interesting to probabilists with the publication of the papers by Harrison and Kreps [15] and Harrison and Pliska [16], who first defined the problem as a martingale representation problem under an equivalent change of measure. Though the diffusion model for the stock price was simpler than our model, this approach is still basically the method we shall be using in this chapter.

Various attempts have been made to generalize the model for the stock price ξ. Using a model that included a diffusion term as well as a jump term given by a Poisson random measure, Merton [21] derived an integro-differential equation that generalized the Black and Scholes equation. Other authors have used Poisson jumps (that is, deterministic jumps at random times; see, eg., Elliott and Kopp [11]), but problems arise when one uses random measures (that is, random jumps at random times). In particular, although the diffusion with jumps considered by Aase in [1] is complete in the mathematical sense, that is, contingent claims have a representation of the form (3.2.3), they are not complete in the financial sense. One problem is that (3.2.5) cannot always be written as an integral with respect

to ξ; another problem is that ϕ^d is not predictable (although it is $\widetilde{\mathcal{P}}$-measurable). The recent work of Schweizer, Föllmer and Sondermann, which we follow closely throughout this chapter, provides a resolution to this problem. Their approach is to produce trading strategies that are optimal in the sense of minimizing the risk associated with the trading strategy in the case where ξ is a martingale, or the 'local risk' (defined below), when ξ is a semimartingale.

2. The Model

In contingent claims analysis there are two differences between our model for ξ and common models for a stock price. First, one normally adds a *discount process*, β (where $1/\beta$ is the bond price), which is a continuous process of bounded variation. Second, since the price of a stock, S, is positive, one uses an exponential model for S. Thus, a natural model would be

$$d\beta_t = -\beta_t r_t dt, \quad \beta_0 = 1;$$

$$\frac{dS_t}{S_{t-}} = m_t dt + a_t dW_t + \int_{\mathbf{R}} f_t(y)(\nu(dt,dy) - H(dy)dt), \quad S_0 = x_0. \tag{2.1}$$

One then works with the discounted price process $\xi = \beta S$, which satisfies

$$\frac{d\xi_t}{\xi_{t-}} = (m_t - r_t)dt + a_t dW_t + \int_{\mathbf{R}} f_t(y)(\nu(dt,dy) - H(dy)dt), \qquad \xi_0 = x_0. \tag{2.2}$$

Here we may assume that m, r, a and f are functions of ξ_-; in particular, this implies that β is allowed to be random, but only through the random changes in S. If S were the vector of all stock prices, this may be reasonable; nevertheless, we

continue to restrict ourselves to real-valued processes. So, in terms of the model of Chapter 3, we have

$$(2.3) \quad \mu_t(x) = x(m_t(x) - r_t(x)), \quad \sigma_t(x) = xa_t(x) \quad \text{and} \quad \gamma_t(x,y) = xf_t(x,y).$$

The use of this model has some interesting ramifications. For one thing, assumptions (A1)–(A3) together with (2.3) imply the following growth condition:

$$(A4) \qquad |\mu_t(x)|^2 + |\sigma_t(x)|^2 + \int_{\mathbf{R}} |\gamma_t(x,y)|^2 H(dy) \le L|x|^2.$$

In addition, ξ has the representation:

$$\xi_t = x_0 \exp\Big\{ \int_0^t \Big(m_s - r_s - \frac{1}{2}a_s^2\Big)ds + \int_0^t a_s dW_s - \int_0^t \int_{\mathbf{R}} f_s(y)H(dy)dt + \int_0^t \int_{\mathbf{R}} \ln(1 + f_s(y))\nu(ds,dy)\Big\}.$$

Note that we must assume that

$$(2.4) \qquad f_t(x,y) > -1 \quad \text{for all} \quad t \in [0,T], \quad x,y \in \mathbf{R}.$$

Integro-differential equations such as those of Corollaries 3.2.5 and 3.2.6 would also look different if we considered V to be a function of S and β instead of ξ (cf. Merton [23]). However, Harrison and Kreps in [15] proved that it is sufficient to assume that $\beta \equiv 1$. Therefore, we shall continue to use the notation of Chapter 3, although we do add (A4) to our assumptions, and we may refer to (2.2) and (2.4) as needed.

3. Non-Redundant Contingent Claims

Throughout this chapter, we follow closely the work of Schweizer and Föllmer and Sondermann. In order to motivate this discussion, let us begin with a definition.

Definition 3.1. A *contingent claim*, H, is a square integrable $\mathcal{F}_T$-measurable random variable.

Remarks 3.2: 1) In this paper, we shall only consider contingent claims of the form $H = c(\xi_T)$. The standard example is that of the European call option which has the form

$$H = c(\xi_T) = (\xi_T - K)^+,$$

where K is the exercise price, and T is the maturity or expiration date. In other words, one pays for the right to buy a given stock at the price K and at time T. If $\xi_T > K$, then one exercises the option, buys the stock, and sells it immediately at the market value, making a profit of $\xi_T - K$. If $\xi_T < K$, then one doesn't exercise the option. The question is, what should one be willing to pay at time $t = 0$, for such a contract, that is, what is the *price* of H, or the value of H at time $t = 0$.

2) The usual notion of arbitrage is that if one can replicate a contingent claim (at time $t = T$) through the judicious trading of a stock and a bond, then the initial values of the two investments must be the same, that is, the price of the contingent claim equals the initial investment in the trading strategy. Such a trading strategy is then called a (riskless) *hedge portfolio*. However, in incomplete markets, an exact hedge is not always possible. In this case, Föllmer and Sondermann [13] first proposed minimizing the risk associated with a trading strategy,

which works nicely when the stock price, ξ, is a martingale as we shall see in Section 3. For the semimartingale case, Schweizer solved these problems using "local risk-minimization" (Section 4).

Let us now introduce some definitions (cf. Föllmer and Sondermann [13]). For technical reasons, we define a *trading strategy*, Φ, to be a pair of processes, $\Phi = (\phi, \psi)$ such that

1) ϕ is predictable,

2) ψ is adapted,

3) $E\Big[\int_0^T \phi_t^2 d\langle\xi\rangle_t + \Big(\int_0^T |\phi_t \mu_t| dt\Big)^2\Big] < \infty$,

4) the process

$$V_t(\Phi) := \phi_t \cdot \xi_t + \psi_t \tag{3.1}$$

has right continuous paths and satisfies $V_t(\Phi) \in L^2(\Omega, P)$, for all $0 \le t \le T$. Here we interpret ϕ_t and ψ_t to be the amount of the stock and bond, respectively, held at time t. Thus, $V_t(\Phi)$, called the *value process*, is just the value of the portfolio at time t. The *gain process* from a trading strategy is defined to be the process $\int_0^t \phi_s d\xi_s$, and the (cumulative) *cost process* for a strategy is

$$C_t(\Phi) := V_t(\Phi) - \int_0^t \phi_s d\xi_s. \tag{3.2}$$

A trading strategy Φ such that $C_t(\Phi) \equiv C_0(\Phi)$, for all $0 \le t \le T$ is called *self-financing*. If, for some contingent claim $H \in L^2(\Omega, P)$ there is a self-financing trading strategy such that $V_T(\Phi) = H$, that is

$$H = C_0 + \int_0^T \phi_s d\xi_s, \qquad P - \text{a.s.},$$

then Φ is called a *riskless hedge portfolio* for H, and H is said to be *attainable*. The *price* of H is then just $V_0(\Phi) = C_0(\Phi)$. (Such trading strategies (ϕ^0, ϕ^i) were discussed in Chapter 1.) In general, however, it is not possible to find such riskless hedge portfolios, and so we need some more definitions.

We say that a trading strategy Φ *generates* a *contingent claim* $H \in L^2(\Omega, P)$ if $V_T(\Phi) = H = C_T(\Phi) + \int_0^T \phi_s d\xi_s$. (Alternatively, we say that Φ is H*-admissible*, and that H itself is *admissible*.) Because the cost process is a random process here, we need some notion of an optimal trading strategy. Thus we define the *risk process*, $R_t(\Phi)$, to be the conditional mean square error process, that is

$$R_t(\Phi) := E[(C_T(\Phi) - C_t(\Phi))^2 \mid \mathcal{F}_t]. \tag{3.3}$$

If $C_t(\Phi)$ is a P-martingale, then we call Φ *mean self-financing*. Note that any process defined by a conditional expectation will be assumed to be right continuous with finite left limits.

LEMMA 3.3. *Suppose a trading strategy $\Phi = (\phi, \psi)$ generates a contingent claim H. Then Φ is mean self-financing if and only if ψ satisfies*

$$\psi_t = E\Big[H - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t\Big] + \int_0^t \phi_s d\xi_s - \phi_t \xi_t. \tag{3.4}$$

Proof: Recall that

$$V_t(\Phi) = \phi_t \cdot \xi_t + \psi_t = C_t(\Phi) + \int_0^t \phi_s d\xi_s. \tag{3.5}$$

Suppose Φ is mean self-financing. Then

$$\begin{aligned} C_t(\Phi) &= E[C_T(\Phi) \mid \mathcal{F}_t] \\ &= E[V_T(\Phi) - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t] \\ &= E[H - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t], \end{aligned}$$

and so

$$\begin{aligned} \psi_t &= C_t(\Phi) + \int_0^t \phi_s d\xi_s - \phi_t \cdot \xi_t \\ &= E\Big[H - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t\Big] + \int_0^t \phi_s d\xi_s - \phi_t \cdot \xi_t, \end{aligned}$$

as desired. On the other hand, if ψ is given by (3.1) then by (3.2),

$$\begin{aligned} C_t(\Phi) &= \psi_t - \int_0^t \phi_s d\xi_s + \phi_t \cdot \xi_t \\ &= E\Big[H - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t\Big] \end{aligned}$$

which is a martingale.

4. Risk Minimization

If we assume that the stock price, ξ, is a martingale, then it is possible to minimize the risk associated with a trading strategy (see, eg. Föllmer and Sondermann [13], or Elliott and Föllmer [8]). In the case where H is attainable, that is, there is a constant C_0, and a trading strategy Φ such that $H = C_0 + \int_0^T \phi_t d\xi_t$, then $V_t(\Phi) = C_0 + \int_0^t \phi_s d\xi_s = \phi_t \xi_t + \psi_t$. The risk is clearly zero for all t, and so Φ is called a riskless hedge portfolio.

More generally, suppose we can write $H = \int_0^T \phi_t d\xi_t + C_T$, where C_T is an $\mathcal{F}_T$-measurable random variable. Then since, ξ is a martingale, we may write

$$V_t(\Phi) := E[H \mid \mathcal{F}_t] = E[C_T] + \int_0^t \phi_s d\xi_s + E[C_T \mid \mathcal{F}_t] - E[C_T],$$

and setting $\psi_t = V_t(\Phi) - \phi_t \cdot \xi_t$, we have a trading strategy $\Phi = (\phi, \psi)$ that generates H. Note that Φ is mean self-financing here, since the cost process is $C_t(\Phi) = E[C_T \mid \mathcal{F}_t]$. Note also that C_T depends on ϕ (with ψ determined by Lemma 3.3). Thus we would like to choose ϕ so as to minimize the remaining squared error risk.

To make this notion precise, let us call a trading strategy Φ' an *admissible continuation of* Φ *at time* t if $\Phi'_s = \Phi_s$ for all $s < t$ and $V_T(\Phi') = V_T(\Phi)$. We say that Φ is *risk-minimizing* if for any $0 \le t \le T$,

$$R_t(\Phi) \le R_t(\Phi')$$

for every admissible continuation Φ' of Φ at time t.

LEMMA 4.1. *(see Lemma 2 in [13]). An admissible risk-minimizing trading strategy is mean self-financing.*

Proof: Consider a strategy $\Phi = (\phi, \psi)$ and a fixed time $0 \le t_0 \le T$. Define ψ' by

$$\psi'_t = \begin{cases} \psi_t & \text{for } 0 \le t < t_0 \\ E\left[V_T(\Phi) - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t\right] + \int_0^t \phi_s d\xi_s - \phi_t \cdot \xi_t & \text{for } t_0 \le t \le T. \end{cases}$$

Then $\Phi' = (\phi, \psi')$ is an admissible continuation of Φ at time t_0 with cost process

$$E\left[V_T(\Phi) - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t\right] = E[C_T(\Phi) \mid \mathcal{F}_t] \qquad \text{for } t_0 \le t \le T,$$

and remaining risk

$$\begin{aligned} R_{t_0}(\Phi') &= E[(C_T(\Phi') - C_{t_0}(\Phi'))^2 \mid \mathcal{F}_{t_0}] \\ &= E[(C_T(\Phi) - E[C_T(\Phi) \mid \mathcal{F}_{t_0}])^2 \mid \mathcal{F}_{t_0}] \\ &= E[(C_T(\Phi) - C_{t_0}(\Phi) + C_{t_0}(\Phi) - E[C_T(\Phi) \mid \mathcal{F}_{t_0}])^2 \mid \mathcal{F}_{t_0}] \\ &= E[(C_T(\Phi) - C_{t_0}(\Phi))^2 \mid \mathcal{F}_{t_0}] - (C_{t_0}(\Phi) - E[C_T(\Phi) \mid \mathcal{F}_{t_0}])^2 \\ &\leq E[(C_T(\Phi) - C_{t_0}(\Phi))^2 \mid \mathcal{F}_{t_0}] \\ &= R_{t_0}(\Phi). \end{aligned}$$

Thus if Φ is risk-minimizing, then $C_{t_0}(\Phi) = E[C_T(\Phi) \mid \mathcal{F}_{t_0}]$ for all $0 \leq t_0 \leq T$, i.e., Φ is mean self-financing. (We have assumed that $C(\Phi)$ is right continuous with finite left limits.) □

The next theorem due to Föllmer and Sondermann provides the main result for this section. Recall that two martingales M and M' are *orthogonal* if their product $M \cdot M'$ is also a martingale.

THEOREM 4.2. *(Föllmer and Sondermann). A trading strategy, Φ, generating H is risk-minimizing if and only if the cost process $C_t(\Phi)$ is orthogonal to ξ. Such a Φ is unique.*

Proof: By Lemma 4.1 we need only consider mean self-financing trading strategies. Since ξ is a martingale, the value process of any such strategy, $V_t(\Phi) = C_t(\Phi) + \int_0^t \phi_s d\xi_s$ must then also be a martingale, so that $V_t(\Phi) = E[H \mid \mathcal{F}_t]$.

Consider the unique Kunita–Watanabe decomposition

$$V_t(\Phi) = E[H \mid \mathcal{F}_t] = E[H] + \int_0^t \phi_s d\xi_s + \Gamma_t, \tag{4.1}$$

where Γ is a martingale orthogonal to the martingale ξ. Now, consider the trading strategy $\Phi = (\phi, \psi)$ where ϕ comes from (4.1) and as usual by Lemma 3.3, $\psi_t = E[H \mid \mathcal{F}_t] - \phi_t \cdot \xi_t$. Then Φ is an H-admissible, mean self-financing strategy with cost process $C_t(\Phi) = \Gamma_t + E[H]$. Let $\Phi' = (\phi', \psi')$ be any admissible continuation of Φ at time t. It follows that the remaining cost for Φ' satisfies

$$\begin{aligned} C_T(\Phi') - C_t(\Phi') &= C_T(\Phi') - C_T(\Phi) + C_T(\Phi) - C_t(\Phi) + C_t(\Phi) - C_t(\Phi') \\ &= \Gamma_T - \Gamma_t + V_t(\Phi) - V_t(\Phi') + \int_t^T (\phi_s - \phi'_s) d\xi_s. \end{aligned}$$

Thus, because Γ and ξ are orthogonal,

$$\begin{aligned} R_t(\Phi') &= E[(C_T(\Phi') - C_t(\Phi'))^2 \mid \mathcal{F}_t] \\ &= E[(\Gamma_T - \Gamma_t)^2 \mid \mathcal{F}_t] + (V_t(\Phi) - V_t(\Phi'))^2 \\ &\quad + E\Big[\int_t^T (\phi_s - \phi'_s)^2 d\langle \xi \rangle_s \mid \mathcal{F}_t\Big] \\ &= R_t(\Phi) + (V_t(\Phi) - V_t(\Phi'))^2 + E\Big[\int_t^T (\phi_s - \phi'_s)^2 d\langle \xi \rangle_s \mid \mathcal{F}_t\Big] \end{aligned}$$

which is minimized when $\phi = \phi'$.

5. Local Risk Minimization

In the case where the price process ξ is a semimartingale rather than a martingale, we use a concept of local risk-minimality first introduced by Schweizer.

One can then prove, analogously to the martingale case, that a trading strategy is locally risk-minimizing if and only if the cost process is orthogonal to the martingale part of ξ. In this section, then, we summarize Schweizer's results, adapting them to our model and thereby simplifying some of the proofs.

Definition 5.1. A trading strategy $\Delta = (\delta, \varepsilon)$ is a *small perturbation* if δ is bounded and $\delta_T > \varepsilon_T = 0$.

Recall that in our model of Chapter 3:

$$d\xi_s = \mu_s(\xi_s)ds + \sigma_s(\xi_s)dW_s + \int_{\mathbf{R}} \gamma_s(\xi_{s-}, y)(\nu(ds, dy) - H(dy)ds)$$

where $\mu_s(x)$ is bounded. Thus the drift component of the gain process, $\int_0^t \delta_s d\xi_s$, associated with Δ has total variation given by $\int_0^T |\delta_s| \cdot |\mu_s(\xi_s)|ds$, which is square integrable; that is, the "systematic gains" from trading Δ are finite. The condition $\delta_T = \varepsilon_T = 0$ implies, firstly, that if Φ generates H, then so does $\Phi + \Delta$. Secondly, suppose we denote the restriction of a small perturbation, Δ, to the subinterval $(s, t]$ of $[0, T]$ by

$$\Delta\big|_{(s,t]} := (\delta \cdot 1_{(s,t]},\ \varepsilon \cdot 1_{[s,t)}). \tag{5.1}$$

The asymmetry is due to the fact that δ is predictable, while ε is merely adapted. Then, such a restricted strategy is also a small perturbation.

In order to discuss local risk minimization, we consider partitions $\tau = \{t_i,\ i = 0, \ldots, N\}$ of the interval $[0, T]$, where $0 = t_0 < t_1 < \cdots < t_N = T$. Denote the mesh of the partition by $\|\tau\| := \max_{1 \le i \le N} \{t_i - t_{i-1}\}$. We shall always assume that any sequence of partitions $\{\tau_n\}_{n=1}^{\infty}$ is *increasing*, that is, $\tau_n \subseteq \tau_{n+1}$ for all n.

Definition 5.2. Let Φ be a trading strategy, Δ a small perturbation, and τ a partition of $[0,T]$. We define the R-quotient (or local risk) $r^\tau[\Phi,\Delta]$ by

$$r^\tau[\Phi,\Delta] = \sum_{t_i\in\tau} \frac{R_{t_i}(\Phi+\Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\Phi)}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]}(w)\cdot 1_{[t_i,t_{i+1}]}(t),$$

where M is the martingale part of ξ. We call the trading strategy Φ *locally risk-minimizing* if

$$\liminf_{n\to\infty} r^{\tau_n}[\Phi,\Delta] \geq 0 \qquad P\times\langle M\rangle \text{-a.s.}$$

for every small perturbation Δ and every increasing sequence $\{\tau_n\}$ of partitions of $[0,T]$ such that $\|\tau_n\| \to 0$.

Remark 5.3: The presence of the denominator in the above definition is clearly necessary, for otherwise, the numerator would approach zero trivially. Considering the local risk relative to the quadratic variation of the price process seems to make the appropriate time scale for these measurements; and conditioning the denominator with respect to $\mathcal{F}_{t_i}$ ensures that the R-quotient is adapted. It should become clear below that the ordinary risk-minimization approach could only be followed through if the risk is defined under an equivalent change of measure, which is not appropriate.

The following result is analogous to Lemma 4.1.

LEMMA 5.4. *(Lemma 2.1 of [27]) If a trading strategy $\Phi = (\phi,\psi)$ is locally risk-minimizing, then it is mean self-financing.*

Proof: Given Φ, construct $\Phi' = (\phi,\psi')$ by setting, for all $0 \leq t \leq T$,

$$\psi'_t = E[V_T(\Phi) - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t] + \int_0^t \phi_s d\xi_s - \phi_t\cdot\xi_t$$

so that Φ' is mean self-financing with cost process $C_t(\Phi') = E[V_T(\Phi) - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t]$ (see Lemma 3.3). Then $\Delta := \Phi' - \Phi = (0, \psi' - \psi)$ is a small perturbation. For any $s < t$,

$$\begin{aligned} V_s(\Phi + \Delta|_{(s,t]}) &= \phi_s \cdot \xi_s + \psi'_s \\ &= V_s(\Phi) + \psi'_s - \psi_s \\ &= V_s(\Phi') \end{aligned}$$

(see equation (3.1)), whence it follows that (since $V_T(\Phi) = V_T(\Phi') = V_T(\Phi + \Delta|_{(s,t]})$),

$$\begin{aligned} &C_T(\Phi + \Delta|_{(s,t]}) - C_s(\Phi + \Delta|_{(s,t]}) \\ &= \left(V_T(\Phi + \Delta|_{(s,t]}) - \int_0^T \phi_r d\xi_r\right) - V_s\left(\phi + \Delta|_{(s,t]}) - \int_0^s \phi_r d\xi_r\right) \\ &= \left(V_T(\Phi') - \int_0^T \phi_r d\xi_r\right) - \left(V_s(\Phi') - \int_0^s \phi_r d\xi_r\right) \\ &= C_T(\Phi') - C_s(\Phi'). \end{aligned}$$

Thus (since $C_T(\Phi') = C_T(\Phi)$),

$$\begin{aligned} R_s(\Phi + \Delta|_{(s,t]}) - R_s(\Phi) &= R_s(\Phi') - R_s(\Phi) \\ &= E[(C_T(\Phi') - C_s(\Phi'))^2 \mid \mathcal{F}_s] - E[(C_T(\Phi) - C_s(\Phi))^2 \mid \mathcal{F}_s] \\ &= E\Big[([C_T(\Phi') - C_s(\Phi')] + [C_T(\Phi) - C_s(\Phi)]) \\ &\quad \times ([C_T(\Phi') - C_s(\Phi')] - [C_T(\Phi) - C_s(\Phi)]) \mid \mathcal{F}_s\Big] \\ &= [C_s(\Phi) - C_s(\Phi')](2E[C_T(\Phi) \mid \mathcal{F}_s] - C_s(\Phi') - C_s(\Phi)) \\ &= -(C_s(\Phi) - E[C_T(\Phi) \mid \mathcal{F}_s])^2, \end{aligned}$$

where the last equality follows from the fact that Φ' is mean self-financing. Therefore, for any partition τ,

$$r^{\tau}[\Phi,\Delta] = -\sum_{t_i\in\tau} \frac{(C_{t_i}(\Phi) - E_T[C_T(\Phi) \mid \mathcal{F}_{t_i}])^2}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i,t_{i+1}]}. \tag{5.2}$$

Now, suppose that for some $t_0 \in [0,T]$, there is a set B of positive probability such that

$$C_{t_0}(\Phi)(\omega) \neq E[C_T(\Phi) \mid \mathcal{F}_{t_0}](\omega)$$

for all $\omega \in B$. Then, since both $C_0(\Phi)$ and $E[C_T(\Phi) \mid \mathcal{F}_\cdot]$ are assumed to be right continuous, there exists for any $\omega \in B$ constants $\delta(\omega) > 0$ and $\varepsilon(\omega) > 0$, so that

$$|C_t(\Phi)(\omega) - E[C_T(\Phi) \mid \mathcal{F}_t](\omega)| \geq \varepsilon(\omega) > 0$$

for all $t \in [t_0,\ t_0 + \delta(\omega)]$. Thus if we construct an increasing sequence of partitions (τ_n) with $\|\tau_n\| \to 0$ such that $t_0 \in \tau_1$, then it follows from (5.2) that

$$\liminf_n r^{\tau_n}[\Phi,\Delta](\omega,t) < 0$$

for any $\omega \in B$ and $t \in (t_0,\ t_0 + \delta(\omega))$ which contradicts the local risk minimality of Φ. Thus, Φ is mean self-financing. □

Before we can prove the main result of this section, we need to establish a technical point. We would like to show that, in order to prove that a trading strategy, Φ, is locally risk-minimizing, we need only consider small perturbations Δ such that $\Phi + \Delta$ is mean self-financing. This will allow us to use martingale techniques later.

Fix a contingent claim H and an H-admissible mean self-financing trading strategy $\Phi = (\phi, \psi)$. Now, for a small perturbation $\Delta = (\delta, \varepsilon)$ and partition τ of $[0,T]$, we want to compare the perturbed trading strategy $\Phi + \Delta\big|_{(t_i,t_{i+1}]} = (\phi + \delta\big|_{(t_i,t_{i+1}]}, \psi + \varepsilon\big|_{(t_i,t_{i+1}]})$ with the corresponding mean self-financing strategy $(\phi + \delta_{(t_i,t_{i+1}]}, \varepsilon')$ where ε' is given by Lemma 3.1. We write $C_t(\Phi + \Delta\big|_{(t_i,t_{i+1}]})$ and $R_t(\Phi + \Delta\big|_{(t_i,t_{i+1}]}$ when we are considering the original small perturbation and $C_t(\phi + \delta\big|_{(t_i,t_{i+1}]})$ and $R_t(\phi + \delta\big|_{(t_i,t_{i+1}]})$ when we are using the small perturbation that preserves the mean self-financing property.

Recall that $C_T(\Phi) = H - \int_0^T \phi_s d\xi_s$ P-a.s., which is independent of the bond portfolio ψ. On the other hand, we have

$$\begin{aligned} C_T(\Phi + \Delta\big|_{(t_i,t_{i+1}]}) &= H - \int_0^T \phi_s d\xi_s - \int_{t_i}^{t_{i+1}} \delta_s d\xi_s \\ &= C_T(\Phi) - \int_{t_i}^{t_{i+1}} \delta_s d\xi_s \\ &= C_T(\phi + \delta\big|_{(t_i,t_{i+1}]}). \end{aligned} \tag{5.3}$$

Moreover, because

$$V_t(\Phi) = C_T(\Phi) + \int_0^t \phi_s d\xi_s = \phi_t \cdot \xi_t + \psi_t,$$

and by definition, $\Delta\big|_{(s,t]} = (\delta \cdot 1_{(s,t]},\ \varepsilon \cdot 1_{[s,t)})$ (equation (5.1)), we have that

$$\begin{aligned} C_{t_i}(\Phi + \Delta\big|_{(t_i,t_{i+1}]}) &= -\int_0^{t_i} \phi_s d\xi_s + \phi_{t_i}\xi_{t_i} + \psi_{t_i} + \varepsilon_{t_i} \\ &= C_{t_i}(\Phi) + \varepsilon_{t_i}, \end{aligned}$$

while the corresponding mean self-financing perturbed strategy satisfies

$$\begin{aligned} C_{t_i}(\phi + \delta\big|_{(t_i,t_{i+1}]}) &= E[C_T(\phi + \delta\big|_{(t_i,t_{i+1}]}) \mid \mathcal{F}_{t_i}] \\ &= E\Big[C_T(\Phi) - \int_{t_i}^{t_{i+1}} \delta_s d\xi_s \mid \mathcal{F}_{t_i}\Big] \\ &= C_{t_i}(\Phi) - E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big] \end{aligned} \tag{5.4}$$

since Φ is mean self-financing. Therefore,

$$\begin{aligned} & C_T(\Phi + \Delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\Phi + \Delta|_{(t_i,t_{i+1}]}) \\ &= C_T(\phi + \delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\Phi) - \varepsilon_{t_i} \\ &= C_T(\phi + \delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\phi + \delta|_{(t_i,t_{i+1}]}) \\ &\quad - \Big(E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big] + \varepsilon_{t_i}\Big), \end{aligned}$$

and so, by the martingale property of $C(\phi + \delta|_{(t_i,t_{i+1}]})$,

$$\begin{aligned} R_{t_i}(\Phi + \Delta|_{(t_i,t_{i+1}]}) &= E[(C_T(\Phi + \Delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\Phi + \Delta|_{(t_i,t_{i+1}]}))^2 \mid \mathcal{F}_{t_i}] \\ &= R_{t_i}(\phi + \delta|_{(t_i,t_{i+1}]}) + \Big(E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big] + \varepsilon_{t_i}\Big)^2. \end{aligned}$$

There, the local risk can be written

$$\begin{aligned} r^\tau[\Phi, \Delta] = & \sum_{t_i + \tau} \frac{R_{t_i}(\phi + \delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\phi)}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i,t_{i+1}]} \\ & + \sum_{t_i \in \tau} \frac{\Big(E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big] + \varepsilon_{t_i}\Big)^2}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i,t_{i+1}]}. \end{aligned} \tag{5.5}$$

We denote the first term on the right-hand side of (5.5) by $r^\tau[\phi, \delta]$, which is consistent with our previous notation.

LEMMA 5.5. *(Lemma 2.2 of [27]) Let H be a contingent claim, and $\Phi = (\phi, \psi)$, an H-admissible trading strategy. Then the following statements are equivalent.*

1) Φ is locally risk-minimizing; and

2) Φ is mean self-financing and

$$\liminf_{n \to \infty} r^{\tau_n}[\phi, \delta] \geq 0 \qquad P \times \langle M\rangle \text{-a.e.},$$

for all small perturbations $\Delta = (\delta, \varepsilon)$ that preserves the mean self-financing property.

Proof: First, we show that 1) implies 2). From Lemma 5.4, we know that Φ must be mean self-financing; thus 2) is just a special case of 1). On the other hand, equation (5.5) implies that $r^{r_n}[\Phi, \Delta] \geq r^{r_n}[\phi, \delta]$, which proves the converse. □

The above lemma allows us to consider only small perturbations that preserve the mean self-financing property, that is, where we evaluate the local risk, we may assume that the cost process under the perturbed strategy, $C_t(\Phi + \Delta|_{(t_i, t_{i+1}]})$, is again a martingale, denoted $C_t(\phi + \delta|_{(t_i, t_{i+1}]})$. This martingale property is used in the following proposition, due to Schweizer. In [26], Schweizer proved, for a general martingale Y instead of $C(\phi)$, that local risk-minimality under such perturbations is equivalent to the orthogonality of Y and M, a result that is nicely analogous to Theorem 3.

To state this another way, let H be a contingent claim, and $\Phi = (\phi, \psi)$ a mean self-financing H-admissible trading strategy. Consider the Kunita–Watanabe decomposition of the cost process $C_T(\Phi)$ with respect to M under P:

$$C_T(\Phi) = C_0(\Phi) + \int_0^T \mu_s^C dM_s + \Gamma_T^C \quad P\text{-a.s.},$$

where M is again the martingale part of ξ and M and Γ^C are othogonal. We want to show that Φ is locally risk-minimizing if and only if $\mu^C \equiv 0$.

One further assumption, that will be needed later, is the following: There exists a constant $k > 0$ such that for any $x \in \mathbf{R}$, $t \in [0, T]$,

(A5) $$\sigma_t^2(x) + \int_{\mathbf{R}} \gamma_t^2(x, y) H(dy) \geq k|x|^2.$$

Let us also add some notation. First, we define the probability measure P_M by

$$dP_M = dP \times \frac{d\langle M\rangle}{E[\langle M\rangle_T]};$$

then E_M will denote expectation under P_M. Second, for a partition τ, we define on $\Omega \times [0,T]$ the σ-algebra

$$\mathcal{P}^\tau := \sigma\big\{D_0 \times \{0\},\ D_i \times (t_i, t_{i+1}] \mid D_0 \in \mathcal{F}_0,\ t_i \in \tau,\ D_i \in \mathcal{F}_{t_i}\big\}.$$

For the remainder of this section we fix an increasing sequence (τ_n) of partitions such that $\lim_{n\to\infty} \|\tau_n\| = 0$. Note that

$$\mathcal{P} = \sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{P}^{\tau_n}\Big),$$

where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times [0,T]$.

The proof of the following proposition is somewhat simpler under our model.

PROPOSITION 5.6. *(cf. Proposition 3.1 of [26]) Under assumptions (A1) – (A5) on ξ, we have*

$$\lim_{n\to\infty} r^{\tau_n}[\phi,\delta] = \phi^2 - 2\phi\cdot\mu^C \quad P\times\lambda \text{ a.s.}, \tag{5.6}$$

for every small perturbation $\Delta = (\delta,\varepsilon)$ and every increasing sequence of partitions (τ_n) such that $\|\tau_n\| \to \infty$. Here λ is Lebesgue measure on $[0,T]$, and $r^{\tau_n}[\phi,\delta]$ is as in Lemma 5.5.

Proof: Note that, because $\langle M\rangle_t = \int_0^t \sigma_s^2(\xi_{s-})ds + \int_0^t \int_{\mathbf{R}} \gamma_s^2(\xi_{s-},y)H(dy)$ assumption (A5) implies that the measures $d\lambda$ and $d\langle M\rangle$ are equivalent; thus it suffices to show that (5.6) holds $P\times\langle M\rangle$-a.s., or P_M-a.s.

From equation (5.3) and (5.4), we know that

$$
\begin{aligned}
&C_T(\phi+\delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\phi+\delta|_{(t_i,t_{i+1}]}) \\
&= C_T(\Phi) - \int_{t_i}^{t_{i+1}} \delta_s d\xi_s - C_{t_i} + E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big] \\
&= C_T(\phi) - C_{t_i}(\phi) - \int_{t_i}^{t_{i+1}} \delta_s dM_s - \Big(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds - E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big]\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&R_{t_i}(C(\phi+\delta|_{(t_i,t_{i+1}]})) - R_{t_i}(C(\phi)) \\
&= E\Big[\Big(\int_{t_i}^{t_{i+1}} \delta_s dM_s\Big)^2 \mid \mathcal{F}_{t_i}\Big] + \mathrm{Var}\Big(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big) \\
&\quad - 2E\Big[(C_T(\phi) - C_{t_i}(\phi))\Big(\int_{t_i}^{t_{i+1}} \delta_s dM_s\Big)|\mathcal{F}_{t_i}\Big] \\
&\quad - 2E\Big[(C_T(\phi) - C_{t_i}(\phi))\Big(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds - E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big]\Big)|\mathcal{F}_{t_i}\Big] \\
&\quad + 2E\Big[\Big(\int_{t_i}^{t_{i+1}} \delta_s dM_s\Big)\Big(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds - E\Big[\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big]\Big)|\mathcal{F}_{t_i}\Big] \\
&= E\Big[\int_{t_i}^{t_{i+1}} \delta_s^2 d\langle M\rangle_s \mid \mathcal{F}_{t_i}\Big] + \mathrm{Var}\Big(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big) \\
&\quad - 2E\Big[(C_{t_{i+1}}(\phi) - C_{t_i}(\phi))\Big(\int_{t_i}^{t_{i+1}} \delta_s dM_s\Big) \mid \mathcal{F}_{t_i}\Big] \\
&\quad + 2\,\mathrm{Cov}\Big(\int_{t_i}^{t_{i+1}} \delta_s dM_s - (C_{t_{i+1}}(\phi) - C_{t_i}(\phi)), \int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big) \\
&= E\Big[\int_{t_i}^{t_{i+1}} (\delta_s^2 - 2\mu_s^c \delta_s) d\langle M\rangle_s \mid \mathcal{F}_{t_i}\Big] \\
&\quad + \mathrm{Var}\Big(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big) \\
&\quad + 2\,\mathrm{Cov}\Big(\int_{t_i}^{t_{i+1}} \delta_s dM_s - (C_{t_{i+1}}(\phi) - C_{t_i}(\phi)), \int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\Big),
\end{aligned}
$$

where the second equality follows from the martingale property of $C(\phi)$ (as in Lemma 5.5) and the third from the Kunita–Watanabe decomposition of $C(\phi)$. It follows that

$$(5.7)\quad r^{\tau_n}[\phi,\delta] = \sum_{t_i\in\tau_n} \frac{E\left[\int_{t_i}^{t_{i+1}}(\delta_s^2 - 2\mu_s^c\delta_s)d\langle M\rangle_s \mid \mathcal{F}_{t_i}\right]}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]}\cdot 1_{(t_i,t_{i+1}]}$$

$$+ \sum_{t_i\in\tau_n} \frac{\mathrm{Var}\left[\int_{t_i}^{t_{i+1}}\delta_s\mu_s ds \mid \mathcal{F}_{t_i}\right]}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]}\cdot 1_{(t_i,t_{i+1}]}$$

$$+2\sum_{t_i+\tau_n} \frac{\mathrm{Cov}\left(\int_{t_i}^{t_{i+1}}\delta_s dM_s - (C_{t_{i+1}}(\phi) - C_{t_i}(\phi)), \int_{t_i}^{t_{i+1}}\delta_s\mu_s ds \mid \mathcal{F}_{t_i}\right)}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]}\cdot 1_{(t_i,t_{i+1}]}.$$

The first term on the right-hand side of (5.7) can be written

$$\sum_{t_i\in\tau_n} \frac{E\left[\int_{t_i}^{t_{i+1}}(\delta_s^2 - 2\mu_s^c\delta_s)d\langle M_s\rangle \mid \mathcal{F}_{t_i}\right]}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]}\cdot 1_{(t_i,t_{i+1}]} = E_M[\delta^2 - 2\mu^c\delta \mid \mathcal{P}^{\tau_n}].$$

To see this, note that the left-hand side of the above equation is $\mathcal{P}^{\tau_n}$-measurable, and let $D_i \in \mathcal{F}_{t_i}$, so that $D_i \times (t_i, t_{i+1}] \in \mathcal{P}^{\tau_n}$. Multiplying the left-hand side by $1_{D_i}\cdot 1_{(t_i,t_{i+1}]}$ and integrating with respect to P_M gives

$$\int \frac{E\left[\int_{t_i}^{t_{i+1}}(\delta_s^2 - 2\mu_s^c\delta_s)d\langle M\rangle_s \mid \mathcal{F}_{t_i}\right]}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]}\cdot 1_{D_i}\cdot 1_{(t_i,t_{i+1}]}dP \times \frac{d\langle M\rangle}{E[\langle M\rangle_T]}$$

$$= E\left[\frac{E\left[\int_{t_i}^{t_{i+1}}(\delta_s^2 - 2\mu_s^c\delta_s)d\langle M\rangle_s \mid \mathcal{F}_{t_i}\right]\cdot 1_{D_i}\cdot[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}]}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i}]E[\langle M\rangle_T]}\right]$$

$$= E\Big[1_{D_i}\cdot\frac{1}{E[\langle M\rangle_T]}\cdot\int_{t_i}^{t_{i+1}}(\delta_s^2 - 2\mu_s^c\delta_s)d\langle M\rangle_s\Big]$$

$$= \int E_M[\delta^2 - 2\mu^c\delta \mid \mathcal{P}^{\tau_n}]\cdot 1_{D_i}\cdot 1_{(t_i,t_{i+1}]}dP \times \frac{d\langle M\rangle}{\langle M\rangle_T},$$

as desired. Now, by martingale convergence, this first term, $E_M[\delta^2 - 2\mu^c\cdot\delta \mid \mathcal{P}^{\tau_n}]$, tends to $\delta^2 - 2\mu^c\cdot\delta$, since this is $\mathcal{P}$-measurable.

The second term in the right-hand side of (5.7) satisfies

$$\sum_{t_i \in \tau_n} \frac{\mathrm{Var}\left(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\right)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}$$

$$\leq \sum_{t_i \in \tau_n} \frac{E\left[\left(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds\right)^2 \mid \mathcal{F}_{t_i}\right]}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}$$

$$\leq \|\delta\|_\infty^2 \cdot \frac{L}{k} \sum_{t_i \in \tau_n} (t_{i+1} - t_i) \cdot 1_{(t_i, t_{i+1}]} \tag{5.8}$$

which converges to zero. Here k is the lower bound of Assumption (A5), and L is from Assumption (A4).

The third term can be separated into two parts, the first part being

$$\sum_{t_i \in \tau_n} \frac{\mathrm{Cov}\left(\int_{t_i}^{t_{i+1}} \delta_s dM_s,\ \int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\right)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}.$$

By the Cauchy–Schwarz inequality, this is dominated by

$$\left(\sum_{t_i \in \tau_n} \frac{E\left[\left(\int_{t_i}^{t_{i+1}} \delta_s dM_s\right)^2 \mid \mathcal{F}_{t_i}\right]}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}\right)^{1/2} \cdot$$

$$\cdot \left(\sum_{t_i \in \tau_n} \frac{\mathrm{Var}\left(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\right)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}\right)^{1/2}$$

$$\leq \|\delta\|_\infty \left(\sum_{t_i \in \tau_n} \frac{\mathrm{Var}\left(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\right)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}\right)^{1/2}$$

which converges to zero by (5.8).

The final term that we need to consider is

$$\sum_{t_i \in \tau_n} \frac{\mathrm{Cov}\left(C_{t_{i+1}}(\phi) - C_{t_i}(\phi), \int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\right)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}$$

$$\leq \left(\sum_{t_i \in \tau_n} \frac{E[(C_{t_{i+1}}(\phi) - C_{t_i}(\phi))^2 \mid \mathcal{F}_{t_i}]}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}\right)^{1/2} \cdot$$

$$\cdot \left(\sum_{t_i \in \tau_n} \frac{\mathrm{Var}\left(\int_{t_i}^{t_{i+1}} \delta_s \mu_s ds \mid \mathcal{F}_{t_i}\right)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}\right)^{1/2} \cdot$$

Thus, again by (5.8) it suffices to show that

$$\sum_{t_i \in \tau_n} \frac{E[(C_{t_{i+1}}(\phi) - C_{t_i}(\phi))^2 \mid \mathcal{F}_{t_i}]}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}$$

$$= \sum_{t_i \in \tau_n} \frac{E[\langle C(\phi) \rangle_{t_{i+1}} - \langle C(\phi) \rangle_{t_i} \mid \mathcal{F}_{t_i}]}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \cdot 1_{(t_i, t_{i+1}]}$$

is bounded. But this is just equal to

$$\frac{E[\langle C(\phi) \rangle_T]}{E[\langle M \rangle_T]} \cdot \frac{dP_C}{dP_M}\Big|_{\mathcal{P}^{\tau_n}},$$

where

$$dP_C = dP \times \frac{d\langle C(\phi) \rangle}{E[\langle C(\phi) \rangle_T]}.$$

This is well defined, since

$$E[\langle M \rangle_T] = E\Big[\int_0^T \sigma_s^2 ds + \int_0^T \int_{\mathbf{R}} \gamma_s^2(y) H(dy) ds\Big] \geq kT$$

by Assumption (A5), and $E[\langle C(\phi) \rangle_T] = E[C(\phi)_T^2] < \infty$ by the definition of a trading strategy. Now, $\frac{dP_\chi}{dP_M}\Big|_{\mathcal{P}^{\tau_n}}$ is a nonnegative $(P_M, \mathcal{P}^{\tau_n})$-supermartingale, and is therefore bounded in n, P_M-a.s. □

We can now use Proposition 5.6 to prove that a trading strategy is locally risk-minimizing if and only if the cost process is orthogonal to the martingale part of ξ.

THEOREM 5.7. *Under assumptions (A1) – (A4) the following statements are equivalent:*

1) $\liminf_{n \to \infty} r^{\tau_n}[\phi, \delta] \geq 0$ $P \times \lambda$ a.e. *for every small perturbation* $\Delta = (\delta, \varepsilon)$ *that preserves the mean self-financing property;*

2) Φ is locally risk minimizing;

3) $\mu^c = 0$ $P \times \lambda$ a.e.;

4) $C(\phi)$ and M are orthogonal.

Proof: (Schweizer [26], Theorem 3.2) The equivalence of (1) and (2) is given by Lemma 5.5. Proposition 5.6 shows that the limit in (1) exists $P \times \lambda$ a.e. and equals $\delta^2 - 2 \cdot \delta \cdot \mu^c$. To prove that (1) implies (3), we choose $\delta := \varepsilon \cdot \text{sgn } \mu^c$ and then let $\varepsilon \to 0$. □

6. Pricing Measures

At this point, our remaining problem is to find the appropriate measure $P^{g,h}$ that should be used to price the contingent claim, where the *price* of a contingent claim, H, is just the initial cost of the trading strategy generating H. Here we restrict ourselves to considering contingent claims of the form $H = c(\xi_T)$, where c is twice differentiable, and $c(\xi_T)$, $c'(\xi_T)$ and $c''(\xi_T)$ are square integrable. Using the measure $P^{g,h}$, we can then use our previous representation results together with Schweizer's results to find the local risk-minimizing hedge portfolio generating $c(\xi_T)$.

To recapitulate Schweizer's conclusions, we want a representation

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_s d\xi_s + \Gamma_t \tag{6.1}$$

such that (i) Γ is a P-martingale (that is, the cost process, denoted by Γ here, is a P-martingale, or ϕ is mean self-financing) and (ii) Γ is orthogonal to the martingale part of ξ under P (by Theorem 5.7). In addition by Harrison and

Kreps [15, Theorem 2], we also know that $P^{g,h}$ must be an equivalent martingale measure, that is, we require (iii) $P^{g,h}$ must be equivalent to P and ξ must be a $P^{g,h}$-martingale. Finally, we add the condition that (iv) $V(\phi)$ must be a $P^{g,h}$-martingale. In the self-financing case, that is, when cost is constant, (iii) implies (iv). Using (iv), we immediately see that

$$V_t(\phi) \equiv E^{g,h}[c(\xi_T) \mid \mathcal{F}_t] =: V(t, \xi_t), \tag{6.2}$$

where $V(t, \xi_t)$ is the process discussed in Chapter 3. Therefore, we can use the results from Chapter 3 to find explicit formulae for the trading strategies and changes of measure required here.

Let us call an equivalent probability measure $P^{g,h}$ a *pricing measure* for the contingent claim $c(\xi_T)$ if it satisfies (i) – (iv). In this case, the price of $c(\xi_T)$ is just $E^{g,h}[c(\xi_T)]$.

We now show that properties (i) – (iv) are sufficient to find a pricing measure (although the measure may not be unique). This pricing measure will then be shown to be equivalent to the *minimal martingale measure* of Schweizer [28], so that our approach is an alternative to his. In the process of finding the pricing measure, we shall also solve for the local risk-minimizing trading strategy generating $c(\xi_T)$.

THEOREM 6.1. *Assume that* $\sigma_t(\xi_t) > 0$ *for all* $0 \le t \le T$ P-a.s. *A pricing measure for* $c(\xi_T)$ *is given by* $P^{g,h}$ *where*

$$h_t(\xi_{t-}, y) = 1 - \frac{\mu_t(\xi_{t-})\gamma_t(\xi_{t-}, y)}{\sigma_t^2(\xi_{t-}) + \int_{\mathbf{R}} \gamma_t^2(\xi_{t-}, y')H(dy')} \tag{6.3}$$

and

$$g_t(\xi_t) = -\frac{\mu_t(\xi_t)\sigma_t(\xi_t)}{\sigma_t^2(\xi_t) + \int_{\mathbf{R}} \gamma_t^2(\xi_{t-}, y')H(dy')} \tag{6.4}$$

which gives a Radon–Nikodym derivative of the form

$$G_t := \frac{dP^{g,h}}{dP}\Big|_{\mathcal{F}_t} = 1 - \int_0^t G_{s-}\Big(\frac{\mu_s(\xi_{s-})}{\sigma_s^2(\xi_{s-}) + \int_{\mathbf{R}} \gamma_s^2(\xi_{s-}, y)H(dy)}\Big)dM_s. \tag{6.5}$$

Here, M is the martingale part of ξ. In addition G_t is square integrable for all $0 \le t \le T$.

This pricing measure is unique if the following equation for h has a unique solution:

$$\begin{aligned} 0 = \int_{\mathbf{R}} &[\phi_t^c(\xi_{t-})\gamma_t(\xi_{t-}, y) - \sigma_t(\xi_{t-})\phi_t^d(\xi_{t-}, y)] \\ &\times \Big[h_t(\xi_{t-}, y) - 1 + \frac{\mu_t(\xi_{t-})\gamma_t(\xi_{t-}, y)}{\sigma_t^2(\xi_{t-}) + \int_{\mathbf{R}} \gamma_t^2(\xi_{t-}, y')H(dy')}\Big]H(dy), \end{aligned} \tag{6.6}$$

where ϕ^c and ϕ^d are given by equations (3.2.6) and (3.2.7), respectively.

Proof: From equations (6.1), (6.2) and (3.2.5), we can write

$$\begin{aligned} \Gamma_t &= -\int_0^t \phi_s d\xi_s + \int_0^t \phi_s^c(\xi_s)(dW_s - g_s(\xi_s))ds \\ &\quad + \int_0^t \int_{\mathbf{R}} \phi_s^d(\xi_{s-}, y)(\nu(ds, dy) - h_s(\xi_{s-}, y)H(dy)ds) \\ &= \int_0^t \phi_s\Big[-\mu_s(\xi_s)ds - \sigma_s(\xi_s)dW_s - \int_{\mathbf{R}} \gamma_s(\xi_{s-}, y)(\nu(ds, dy) - H(dy)ds)\Big] \\ &\quad + \int_0^t \phi_s^c(\xi_s)dW_s - \int_0^t \phi_s^c(\xi_s)g_s(\xi_s)dr \\ &\quad + \int_0^t \int_{\mathbf{R}} \phi_s^d(\xi_{s-}, y)(\nu(ds, dy) - H(dy)ds) \end{aligned}$$

$$+\int_0^t\int_{\mathbf{R}}\phi_s^d(\xi_{s-},y)H(dy)ds-\int_0^t\int_{\mathbf{R}}\phi_s^d(\xi_{s-},y)h_s(\xi_{s-},y)H(dy)dr$$

$$=\int_0^t[-\phi_s\sigma_s(\xi_s)+\phi_s^c(\xi_s)]dW_s$$

$$+\int_0^t\int_{\mathbf{R}}[-\phi_s\gamma_s(\xi_{s-},y)+\phi_s^d(\xi_{s-},y)](\nu(ds,dy)-H(dy)ds)$$

$$+\int_0^t\Big\{-\phi_s\mu_s(\xi_s)-\phi_s^c(\xi_s)g_s(\xi_s)+\int_{\mathbf{R}}\phi_s^d(\xi_{s-},y)(1-h_s(\xi_{s-},y))H(dy)\Big\}ds.$$

Thus, Γ is a P-martingale (that is, ϕ is mean self-financing) if and only if

$$-\phi_s\mu_s(\xi_s)-\phi_s^c(\xi_s)g_s(\xi_s)+\int_{\mathbf{R}}\phi_s^d(\xi_{s-},y)(1-h_s(\xi_{s-},y))H(dy)=0 \tag{6.7}$$

for all $0\le s\le T$, P-almost surely.

Next, we use the condition that Γ and M must be orthogonal under P. This is equivalent to requiring that the quadratic covariation, $[\Gamma,M]$, be a martingale under P. Now, since $M^c=\xi^c$ and $\Delta M=\Delta\xi$,

$$[\Gamma,M]_t=\langle\Gamma^c,\xi^c\rangle_t+\sum_{0\le s\le t}\Delta\Gamma_s\Delta\xi_s$$

$$=\int_0^t\sigma_s(\xi_s)(-\phi_s\sigma_s(\xi_s)+\phi_s^c(\xi_s))ds$$

$$+\int_0^t\int_{\mathbf{R}}\gamma_s(\xi_{s-},y)(-\phi_s\cdot\gamma_s(\xi_{s-},y)+\phi_s^d(\xi_{s-},y))\nu(ds,dy)$$

$$=\int_0^t\int_{\mathbf{R}}\gamma_s(\xi_{s-},y)(-\phi_s\cdot\gamma_s(\xi_{s-},y)+\phi_s^d(\xi_{s-},y))(\nu(ds,dy)-H(dy)ds)$$

$$+\int_0^t\Big\{\sigma_s(\xi_s)(-\phi_s\sigma_s(\xi_s)+\phi_s^c(\xi_s))$$

$$+\int_{\mathbf{R}}\gamma_s(\xi_{s-},y)(-\phi_s\gamma_s(\xi_{s-},y)+\phi_s^d(\xi_{s-},y))H(dy)\Big\}ds.$$

It follows that $[\Gamma, M]$ is a P-martingale if and only if

(6.8)

$$\sigma_s(\xi_s)(-\phi_s\sigma_s(\xi_s) + \phi_s^c(\xi_s)) + \int_{\mathbb{R}} \gamma_s(\xi_{s-}, y)(-\phi_s\gamma_s(\xi_{s-}, y) + \phi_s^d(\xi_{s-}, y))H(dy) = 0$$

for all $0 \leq s \leq T$, P-a.s.

The final condition that we require is that ξ be a $P^{g,h}$-martingale. Rewriting equation (3.1.1) we find that

$$\begin{aligned}\xi_{0,t}(x_0) = \xi_t = x_0 &+ \int_0^t \sigma_s(\xi_s)(dW_s - g_s(\xi_s)ds) \\ &+ \int_0^t \int_{\mathbb{R}} \gamma_s(\xi_{s-}, y)(\nu(ds, dy) - h_s(\xi_{s-}, y)H(dy)ds) \\ &+ \int_0^t \left\{\mu_s(\xi_s) + \sigma_s(\xi_s)g_s(\xi_s) + \int_{\mathbb{R}} \gamma_s(\xi_{s-}, y)(h_s(\xi_{s-}, y) - 1)H(dy)\right\}ds.\end{aligned}$$

Therefore, ξ is a $P^{g,h}$-martingale if and only if

$$\mu_s(\xi_s) + \sigma_s(\xi_s)g_s(\xi_s) + \int_{\mathbb{R}} \gamma_s(\xi_{s-}, y)(h_s(\xi_{s-}, y) - 1)H(dy) = 0 \tag{6.9}$$

for all $0 \leq s \leq T$ P-a.s.

We can immediately solve equation (6.8) for ϕ, giving

$$\phi_s = \frac{\sigma_s(\xi_{s-})\phi_s^c(\xi_{s-}) + \int_{\mathbb{R}} \gamma_s(\xi_{s-}, y)\phi_s^d(\xi_{s-}, y)H(dy)}{\sigma_s^2(\xi_{s-}) + \int_{\mathbb{R}} \gamma_s^2(\xi_{s-}, y')H(dy')} \tag{6.10}$$

which is well-defined since by Assumption (A5) of Section 5, the denominator is strictly positive. Note that ϕ depends on the change of measure through ϕ^c and ϕ^d. (See equations (3.2.6) and (3.2.7) of Theorem 3.2.1.) Now, we multiply equation (6.7) by σ and equation (6.9) by ϕ^c, giving (in abbreviated notation),

$$0 = -\sigma_s\phi_s\mu_s - \sigma_s\phi_s^c g_s + \sigma_s \int_{\mathbb{R}} \phi_s^d(y)(1 - h_s(y))H(dy)$$

and

$$0 = \phi_s^c \mu_s + \sigma_s \phi_s^c g_s + \phi_s^c \int_{\mathbf{R}} \gamma_s(y)(h_s(y) - 1) H(dy).$$

Thus we can eliminate the $\sigma \phi^c g$ term to give

$$\begin{aligned}
0 &= \mu_s(\phi_s^c - \sigma_s \phi_s) + \int_{\mathbf{R}} (h_s(y) - 1)(\phi_s^c \gamma_s(y) - \sigma_s \phi_s^d(y)) H(dy) \\
&= \mu_s \phi_s^c - \frac{\mu_s \sigma_s (\sigma_s \phi_s^c + \int_{\mathbf{R}} \gamma_s(y) \phi_s^d(y) H(dy))}{\sigma_s^2 + \int_{\mathbf{R}} \gamma_s^2(y') H(dy')} \\
&\quad + \int_{\mathbf{R}} (h_s(y) - 1)(\phi_s^c \gamma_s(y) - \sigma_s \phi_s^d(y)) H(dy) \\
&= \frac{\int_{\mathbf{R}} \mu_s \gamma_s(y)(\phi_s^c \gamma_s(y) - \sigma_s \phi_s^d(y)) H(dy)}{\sigma_s^2 + \int_{\mathbf{R}} \gamma_s^2(y') H(dy')} \\
&\quad + \int_{\mathbf{R}} (h_s(y) - 1)(\phi_s^c \gamma_s(y) - \sigma_s \phi_s^d(y)) H(dy) \\
&= \int_{\mathbf{R}} [\phi_s^c \gamma_s(y) - \sigma_s \phi_s^d(y)] \Big[h_s(y) - 1 + \frac{\mu_s \gamma_s(y)}{\sigma_s^2 + \int_{\mathbf{R}} \gamma_s^2(y') H(dy')} \Big] H(dy),
\end{aligned}$$

where the second equality follows from equation (6.10). This proves the second part of the theorem (including equation (6.6)), and the fact that

$$h_s(\xi_{s-}, y) \doteq 1 - \frac{\mu_s(\xi_{s-}) \gamma_s(\xi_{s-}, y)}{\sigma_s^2(\xi_{s-}) + \int_{\mathbf{R}} \gamma_s^2(\xi_{s-}, y') H(dy')}$$

is a solution. Substituting h into equation (6.9) gives the equation for g. Finally, the equation for G follows easily from the definition. (See equation (3.2.1).) □

We discuss the properties of G below.

COROLLARY 6.2. *Suppose $\sigma_t(\xi_t) = 0$ for some $t \in [0, T]$ with positive probability. Then a pricing measure for $c(\xi_T)$ is given by $P^{g,h}$, where $g_t(\xi_t) = 0$ and*

$$h_t(\xi_{t-}, y) = 1 - \frac{\mu_t(\xi_{t-}) \gamma_t(\xi_{t-}, y)}{\int_{\mathbf{R}} \gamma_t^2(\xi_{t-}, y) H(dy)}$$

for all t such that $\sigma_t(\xi_t) = 0$; g and h satisfy (6.3) and (6.4) otherwise. This measure is unique if the following equations have unique solutions for h:

$$(6.11) \qquad \int_{\mathbf{R}} \phi_t^d(\xi_{t-}, y)\Big[h_t(\xi_{t-}, y) - 1 + \frac{\mu_t(\xi_{t-})\gamma_t(\xi_{t-}, y)}{\int_{\mathbf{R}} \gamma_t^2(\xi_{t-}, y')H(dy')}\Big] H(dy) = 0$$

for all t such that $\sigma_t(\xi_{t-}) = 0$; (6.6) otherwise.

Proof: First, note that regardless of our choice of g and h, from equation (3.2.6) we know that $\phi_t^c(\xi_t) = \dfrac{\partial E^{g,h}[c(\xi_T) \mid \xi_t]}{\partial \xi_t} \cdot \sigma_t(\xi_t) = 0$, if $\sigma_t(\xi_t) = 0$. Thus, equations (6.7) – (6.9) become

$$(6.7') \qquad -\phi_s\mu_s + \int_{\mathbf{R}} \phi_s^d(y)(1 - h_s(y))H(dy) = 0$$

$$(6.8') \qquad \int_{\mathbf{R}} \gamma_s(y)(-\phi_s\gamma_s(y) + \phi_s^d(y))H(dy) = 0$$

$$(6.9') \qquad \mu_s + \int_{\mathbf{R}} \gamma_s(y)(h_s(y) - 1)H(dy) = 0.$$

Again, from (6.8′) we can solve for ϕ:

$$\phi_s = \frac{\int_{\mathbf{R}} \gamma_s(y)\phi_s^d(y)H(dy)}{\int_{\mathbf{R}} \gamma_s^2(y)H(dy)}$$

as expected. Substituting this equation for ϕ into (6.7′) gives

$$\frac{-\mu_s \int_{\mathbf{R}} \gamma_s(y)\phi_s^d(y)H(dy)}{\int_{\mathbf{R}} \gamma_s^2(y')H(dy')} + \int_{\mathbf{R}} \phi_s^d(y)(1 - h_s(y))H(dy)$$

$$= -\int_{\mathbf{R}} \Big[h_s(y) - 1 + \frac{\mu_s\gamma_s(y)}{\int_{\mathbf{R}} \gamma_s^2(y')H(dy')}\Big] H(dy) = 0,$$

which establishes a solution for h (it is easily checked that h also satisfies equation (6.9′)), and also gives a sufficient condition for the uniqueness of h. □

COROLLARY 6.3. *A locally risk-minimizing trading strategy $\Phi = (\phi, \psi)$ is generating the contingent claim $c(\xi_T)$ given by*

$$\phi_t = \frac{\sigma_t(\xi_{t-})\phi_t^c(\xi_{t-}) + \int_{\mathbf{R}} \gamma_t(\xi_{t-}, y)\phi_t^d(\xi_{t-}, y)H(dy)}{\sigma_t^2(\xi_{t-}) + \int_{\mathbf{R}} \gamma_t^2(\xi_{t-}, y)H(dy)}$$

and

$$\psi_t = E\Big[c(\xi_T) - \int_0^T \phi_s d\xi_s \mid \mathcal{F}_t\Big] + \int_0^t \phi_s d\xi_s - \phi_t \xi_t.$$

Here ϕ^c and ϕ^d are given by equations (3.2.6) and (3.2.7), using the measure $P^{g,h}$ given by the above theorem.

Proof: It remains to show that ϕ satisfies conditions 1)–4) of the definition of a trading strategy (see Section 3). Clearly ϕ is predictable. From Lemma 3.2.4 we have $\phi^c = \frac{\partial V}{\partial x} \cdot \sigma$ and $\phi^d = V(t, \xi_{t-} + \gamma) - V(t, \xi_{t-})$, so that

$$\int_0^T \phi_t^2 d\langle \xi \rangle_t = \int_0^T \frac{\left\{\sigma^2 \frac{\partial V}{\partial x} + \int_{\mathbf{R}} \gamma^2 \Big[\frac{V(t,\xi_{t-}+\gamma) - V(t,\xi_{t-})}{\gamma}\Big] H(dy)\right\}^2}{\sigma^2 + \int_{\mathbf{R}} \gamma^2 H(dy)} dt.$$

Now, from equation (3.2.8), and the assumption that c' is bounded, and that $\frac{\partial G}{\partial x}$ exists in the mean square sense,

$$\frac{\partial V}{\partial x} = E\Big[\frac{\partial G_{t,T}(x)}{\partial x} \cdot C(\xi_{t,T}(x)) + G_{t,T}(x) \cdot \frac{dc(\xi_{t,T}(x))}{d\xi} \cdot \frac{d\xi_{t,T}(x)}{dx}\Big]$$

$$\leq \text{const.}\ [1 + E[\xi_{t,T}^2(x)]),$$

if $E\Big[\frac{\partial G^2}{\partial x}\Big]$, $E[G_{t,T}^2(x)]$ and $E\Big[\frac{d\xi_{t,T}^2(x)}{dx}\Big]$ are bounded (with upper bound independent of t or x). (Conditions under which this is true are given in Lemmas 6.6 and 6.7 and Proposition 6.9.)

Note that $\frac{V(t,\xi_{t-}+\gamma)-V(t,\xi_{t-})}{\gamma}$ has the same upper bound. It follows that

$$E\int_0^T \phi_t^2 d\langle\xi\rangle_t \leq \text{const.} \int_0^T E\Big[\frac{|\xi_t|^2 + E[\xi_T^2 \mid \xi_t]}{|\xi_t|^2}\Big] ds$$

$$\leq \text{const.}\ T\{1 + E[\xi_T^4] \sup_t E[\xi_t^{-4}]\}.$$

Again, conditions under which this is finite are given in Lemma 6.6. The proof for $E\Big[\Big(\int_0^T |\phi\mu|dt\Big)^2\Big]$ is similar. Finally, $V \in L^2(P)$ if

$$E[V^2] = E\big[\big(E[G_{t,T}(\xi_t)\cdot c(\xi_{t,T}(\xi_t))]\big)^2\big]$$

$$\leq E[G_{t,T}^2(\xi_t)c^2(\xi_T(x_0))] < \infty$$

which holds if $E[G_T^4]$ and $E[\xi_T^4]$ are bounded (see below).

Note that the local risk-minimizing hedge portfolio ϕ depends on the pricing measure (through ϕ^c and ϕ^d), and so ϕ will be uniquely determined if the pricing measure is.

Note also that, although Γ and ξ are both martingales under $P^{g,h}$, Γ is not orthogonal to ξ under $P^{g,h}$. To see this, we modify equation (6.8) to conclude that Γ is orthogonal to ξ under $P^{g,h}$ if and only if

$$\sigma_s(-\hat{\phi}_s\sigma_s + \phi_s^c) + \int_{\mathbb{R}} \gamma_s(y)(-\hat{\phi}_s\gamma_s(y) + \phi_s^d(y))h_s(y)H(dy) = 0,$$

which gives us the portfolio

$$\hat{\phi}_s = \frac{\sigma_s\phi_s^c + \int_{\mathbb{R}} \gamma_s(y)\phi_s^d h_s H(dy)}{\sigma^2 + \int_{\mathbb{R}} \gamma_s^2(y)h_s H(dy)} \neq \phi_s.$$

If we were to choose the portfolio $\hat{\phi}$, that would lead to a different equivalent probability measure, say $P^{\hat{g},\hat{h}}$, and this portfolio would minimize the "risk" under the new measure, that is, $\hat{\phi}$ would minimize $R^{\hat{g},\hat{h}}(\phi) = E^{\hat{g},\hat{h}}[(C_T(\phi) - C_t(\phi))^2 \mid \mathcal{F}_t]$, (see Section 4), which is not appropriate. (This does, however, lead to the generalized minimal martingale measure of Theorem 4.1 of [28].)

The pricing measure given by equation (6.5) is related to a result of Schweizer [28]. To see Schweizer's approach to this problem, first recall that an *equivalent martingale measure* is a probability measure P^*, equivalent to P, such that

$$\frac{dP^*}{dP} \in L^2(\Omega, P),$$

and ξ is a martingale under P^*. A *minimal martingale measure*, $\widehat{P}$, is defined to be an equivalent martingale measure such that $\widehat{P} = P$ on $\mathcal{F}_0$, and any square-integrable martingale that is orthogonal to M under P is still a martingale under $\widehat{P}$. In particular, referring to equation (6.1), if Γ is a P-martingale that is orthogonal to M under P, then Γ and so V are also $\widehat{P}$-martingales. Thus $\widehat{P}$ is indeed a pricing measure.

In general, if $\xi = \xi_0 + M + A$, then put $\alpha = \frac{dA}{d\langle M \rangle}$, assuming this is well-defined. In our model, $A_t = \int_0^t \mu_s(\xi_{s-})ds$, so that

$$\alpha_s = \frac{\mu_s(\xi_{s-})}{\sigma_s^2 + \int_{\mathbf{R}} \gamma_s^2(y)H(dy)}. \tag{6.12}$$

We state the following result without proof.

THEOREM 6.4. *(Schweizer [28], Theorem 2.1)*

1) The minimal martingale measure $\widehat{P}$ is uniquely determined;

2) $\widehat{P}$ exists if and only if the solution, G, of the stochastic differential equation

$$G_t = 1 - \int_0^t G_{s-}\alpha_s dM_s$$

is a strictly positive square-integrable martingale under P; in that case, $\widehat{P}$ is given by

$$\frac{d\widehat{P}}{dP}\Big|_{\mathcal{F}_t} = G_t.$$

Thus, by (6.12) and (6.5), the above theorem shows that our pricing measure is just the minimal martingale measure corresponding to our model. The difference between our approaches is that, on the one hand, the minimal martingale measure satisfies the property that *for any* square integrable P-martingale L that is orthogonal to M, L is a $\widehat{P}$-martingale. Our pricing measure, $P^{g,h}$, on the other hand, simply requires that, for a given trading strategy Φ generating a contingent claim $c(\xi_T)$, the cost process, $\Gamma_t = C_t(\Phi)$, must be a P-martingale, orthogonal to M under P, that is also a $P^{g,h}$-martingale. Thus, (under our model), the minimal martingale measure is a pricing measure *for any* contingent claim of the form $c(\xi_T)$.

For the converse, if the class of all such contingent claims is sufficiently rich, then it is possible that there is a unique probability measure $\widehat{P}$ such that $\widehat{P}$ is a pricing measure for every contingent claim. Consider equation (6.6) once again. From Lemma 3.2.4, we can write

$$\phi_s^c\gamma_s - \sigma_s\phi_s^d = \sigma_s \cdot \gamma_s \cdot \left[\frac{\partial V}{\partial x} - \frac{V(\xi_{s-} + \gamma_s(y)) - V(\xi_{s-})}{\gamma_s(y)}\right].$$

So, formally, assuming that $\sigma, \gamma > 0$, we would need the collection of functions

$$\frac{\partial E^{g,h}[c(\xi_{t,T}(x))]}{\partial x} \quad \frac{E^{g,h}[c(\xi_{t,T}(x+\Delta x))] - E^{g,h}[c(\xi_{t,T}(x))]}{\Delta x}$$

to be complete with respect to the measure H.

Remark 6.5. We now provide conditions under which $\frac{\partial G}{\partial x}$ exists, where the derivative is understood as the limit of the corresponding quotient in the mean square sense, G is the above minimal martingale measure, and x is the initial value of ξ.

First, we require that h be strictly positive, that is

$$1 > \frac{\mu_t(\xi_{t-})\gamma_t(\xi_{t-}, y)}{\sigma_t^2(\xi_{t-}) + \int_{\mathbf{R}} \gamma_t^2(\xi_{t-}, y')H(dy')}.$$

This implies that γ is bounded (in y), and so, adding extra integrability conditions for γ should be reasonable (see below). In addition, from the exponential model ($\mu(x) = x \cdot m(x)$, $\sigma(x) = x - a(x)$ and $\gamma(x) = x \cdot f(x)$ as in section 2), we must have $f > -1$. In fact, we assume that f is bounded away from -1, that is, there exists an $\varepsilon > 0$ such that

$$f_t(x, y) \geq -1 + \varepsilon \quad \text{for all } t, x, y. \tag{6.13}$$

We shall need the following lemma.

LEMMA 6.6. *(i) Using the exponential model, if (6.13) holds, and* $\int_{\mathbf{R}} f^m(x, y)H(dy)$ *is bounded for* $m = 1, \ldots, 2p$, *then*

$$\sup_t E[\xi_{0,t}^{-2p}(x_0)] \leq x_0^{-2p} e^{\text{const. } T},$$

and

$$\sup_t E[\xi_{0,t}^{2p}(x_0)] \le x_0^{2p} e^{\text{const. } T},$$

for any $x_0 \in \mathbf{R}$, where the constants are independent of t or x_0.

(ii) In terms of the original model,

$$\Big| \int_{\mathbf{R}} \frac{\partial \gamma_t^m}{\partial x}(x,y) H(dy) \Big| \le \text{const.} \quad \text{for } m = 1, \dots, 2p,$$

then

$$\sup_{t,x} E\Big[\frac{d\xi_{t,T}^{2p}(x)}{dx}\Big] < \infty.$$

Proof: (i) Under the exponential model, we have

$$\xi_t^{-2p}(x_0) = x_0^{-2p} - 2p \int_0^t \xi_{s-}^{-2p} \Big[m_s ds + a_s dW_s + \int_{\mathbf{R}} f_s(y)(\nu(ds,dy) - H(dy)ds) \Big]$$

(6.14)

$$+ p(2p+1) \int_0^t \xi_{s-}^{-2p} a_s^2 ds$$

$$+ \int_0^t \int_{\mathbf{R}} \xi_{s-}^{-2p} \big[(1 + f_s(\xi_{s-}, y))^{-2p} - 1 + 2p f_s(\xi_{s-}, y) \big] \nu(ds, dy).$$

However,

$$\int_{\mathbf{R}} \big[(1 + f_s(\xi_{s-}, y))^{-2p} - 1 + 2p f_s(\xi_s, y) \big] H(dy)$$

$$= \int_{\mathbf{R}} \Big[\frac{1 - (1 + f_s(\xi_{s-}, y))^{2p}}{(1 + f_s(\xi_{s-}, y))^{2p}} + 2p f_s(\xi_s, y) \Big] H(dy)$$

is bounded by the hypotheses.

For each $N = 1, 2, \dots$, let F^N be a function such that $F^N(x) = x$ for $|x| \le N$, $F^N(x) = 0$ for $|x| \ge N + 1$, and F^N has bounded derivative. Now write

$(\xi_t^N(x_0))^{-2p}$ for the solution of the stochastic differential equation corresponding to (6.14) with ξ_{s-} replaced by $F^N(\xi_{s-}^N)$ on the right-hand side of the equation. Then, for this "truncated" version of (6.14), all the coefficients are bounded (since m and a are bounded by assumption) and so we can take expectations, giving

$$E[F^N(\xi_t^N(x_0))^{-2p}] \leq E[|\xi_t^N(x_0)|^{-2p}]$$

$$\leq x_0^{-2p} + \text{const.} \int_0^t E[F^N(\xi_t^N(x_0))^{-2p}]ds,$$

where the constant is independent of t (and x_0). It thus follows from Gronwall's inequality and Fatou's lemma that $E[\xi_t(x_0)^{-2p}] \leq x_0^{-2p} e^{\text{const. } T}$, as desired. The proof for $E[\xi_{0,t}^{2p}(x_0)]$ is similar.

(ii) Under the original model, we have (see, eg. equation (13), p. 282 of [14]),

$$\frac{d\xi_{t,T}(x)}{dx} = 1 + \int_t^T \mu_s' \frac{d\xi}{dx} ds + \int_t^T \sigma_s' \frac{d\xi}{dx} dW_s$$

$$+ \int_t^T \int_{\mathbf{R}} \frac{\partial \gamma}{\partial x}(\xi_{s-}, y) \frac{d\xi}{dx} (\nu(ds, dy) - H(dy)ds),$$

so that

$$\left[\frac{d\xi_{t,T}(x)}{dx}\right]^{2p} = 1 + 2p \int_t^T \left(\frac{d\xi}{dx}\right)^{2p} \mu' ds + 2p \int_t^T \left(\frac{d\xi}{dx}\right)^{2p} \sigma' dW_s$$

$$+ 2p \int_t^T \int_{\mathbf{R}} \left(\frac{d\xi}{dx}\right)^{2p} \frac{\partial \gamma}{\partial x} (\nu(ds, dy) - H(dy)ds)$$

$$\times \frac{2p(2p-1)}{2} \int_t^T \left(\frac{d\xi}{dx}\right)^{2p} (\sigma')^2 ds$$

$$+ \int_t^T \int_{\mathbf{R}} \left(\frac{d\xi}{dx}\right)^{2p} \left[\left(1 + \frac{\partial \gamma}{\partial x}\right)^4 - 1 - 4\frac{\partial \gamma}{\partial x}\right] (\nu(ds, dy).$$

The remainder of the proof is much like that of part (i), and uses the fact that μ' and σ' are also bounded. Clearly, the upper bound for $E\left[\left(\frac{d\xi_{t,T}(x)}{dx}\right)^{2p}\right]$ depends only on the bounds of μ', σ' and $\int_{\mathbf{R}}\left(\frac{\partial\gamma}{\partial x}(x,y)\right)^m H(dy)$, $m = 1,\ldots,2p$.

Recall that G satisfies the stochastic differential equation

$$(6.15)\quad G_t(x_0) = 1 - \int_0^t G_{s-}\alpha_s\sigma_s dW_s - \int_0^t\int_{\mathbf{R}} G_{s-}\alpha_s\gamma_s(y)(\nu(ds,dy) - H(dy)ds),$$

where $\alpha_s := \frac{\mu_s}{\sigma_s^2 + \int_{\mathbf{R}}\gamma_s^2 H(dy)}$. It can be easily checked, using the assumptions of Section 2 for the coefficients μ, σ, and γ, that $\alpha_s\cdot\sigma_s$ and $\int\alpha_s\gamma_s(y)H(dy)$ are bounded. However, the most we can say about their derivatives is that

$$\left|\frac{d}{dx}\alpha_s(x)\sigma_s(x)\right| + \left|\int_{\mathbf{R}}\frac{d}{dx}\alpha_s(x)\gamma_s(x,y)H(dy)\right| \le \frac{\text{const.}}{|x|},$$

(assuming that $\frac{d}{dx}\int\gamma_s^2(x,y)H(dy) = 2\int\gamma_s(x,y)\gamma_s'(x,y)H(dy)$). Nevertheless, we can establish the existence of $\frac{\partial G}{\partial x}$. First we provide conditions for the integrability of G.

LEMMA 6.7. *If* $|\int_{\mathbf{R}}\gamma^m(x,y)H(dy)| \le$ *const.* $|x|^m$ *for* $m = 1,\ldots,2p$, *then* $\sup_t E[G_t^{2p}(x_s)] < \infty$.

Proof: By Ito's Lemma,

$$\begin{aligned} G_t^{2p}(x_s) = 1 &- 2p\int_0^t G_{s-}^{2p-1}\Big[\alpha_s\sigma_s dW_s - \int_{\mathbf{R}}\alpha_s\gamma_s(y)(\nu(ds,dy) - H(dy)dt) \\ &+ 2p(p-1)\int_0^t G_{s-}^{2p-2}\alpha_s^2\sigma_s^2 ds \\ &+ \int_0^t\int_{\mathbf{R}} G_{s-}^{2p}\left[(1+\alpha_s\gamma_s(y))^{2p} - 1 - 2p\alpha_s\gamma_s(y)\right]\nu(ds,dy). \end{aligned}$$

Using induction, the proof proceeds much like the proof of Lemma 6.6.

Note that the proof of this lemma can also be used to show that for any stopping time $0 \le \tau \le T$, $\sup_\tau E[G_\tau^{2p}(x_0)] < \infty$, so that G is a square integrable martingale.

LEMMA 6.8. *The derivative* $\frac{\partial G_s(x_0)}{\partial x}(\omega)$ *exists for almost every* $\omega \in \Omega$.

Proof: Let us truncate the coefficients of (6.15) as follows: Let F^N be as before, i.e., $F^N(x) = x$ if $|x| \leq N$, $F^N(x) = 0$ if $|x| \geq N+1$, and F^N has finite derivative. Next, define $\alpha^N \sigma^N$ and $\alpha^N \gamma^N$ so that, eg., $\alpha^N(x)\sigma^N(x) = \alpha(x)\sigma(x)$ for $|x| \geq \frac{1}{N}$, but $\alpha^N(x)\sigma^N(x)$ is bounded, with bounded derivative for $|x| \leq \frac{1}{N+1}$, (similarly for $\int_{\mathbf{R}} \alpha^N(x)\gamma^N(x,y)H(dy)$). Then the process

$$G_t^N(x_0) := 1 - \int_0^t F^N(G_{s-}^N)\alpha_s^N \sigma_s^N dW_s$$
$$- \int_0^t \int_{\mathbf{R}} F^N(G_{s-}^N)\alpha_s^N \gamma_s^N(y)(\nu(ds, dy - H(dy)ds)$$

is differentiable except in a set E of probability zero (by Theorem 36.9 of [22]). However, $G(\omega)$ is bounded (since it is a martingale) and $\xi(\omega) > 0$ (see Remark 13.16 of [7]) except on a set E' of probability zero. Thus, for $\omega \notin E \cup E'$, there is an $N_0(\omega)$ such that $G_t^N(x_0)(\omega) = G_t(x_0)(\omega)$ for all $N \geq N_0(\omega)$; thus $\frac{\partial G}{\partial x}$ exists in the almost sure sense. □

We are now ready to prove that $\frac{\partial G}{\partial x}$ exists in the mean-square sense. Recall that this is needed in order to assure that $\frac{\partial V}{\partial x}(t,x)$ is well defined.

PROPOSITION 6.9. *Assume that* $\left|\int_{\mathbf{R}} \gamma_t^m(x,y)H(dy)\right| \leq C|x|^m$, *for* $m = 1,2,3,4$. *Then* $\frac{\partial G}{\partial x}$ *exists in the mean-square sense. In addition,* $\sup_{t,x} E\left[\frac{\partial G_t^2(x)}{\partial x}\right] < \infty$.

Proof: Write $\zeta_t = \zeta_t(x, \Delta x) = \frac{G_t(x+\Delta x) - G_t(x)}{\Delta x}$ where $x, \Delta x \in \mathbf{R}$. By Lemma 6.8, it suffices to show that $|\zeta_t(x,\Delta x)|^2$ is uniformly integrable, which is

true if $\sup_{t,\Delta x} E|\zeta_t(x,\Delta x)|^4 < \infty$. Note that

$$\zeta_t(x,\Delta x) = -\int_0^t \{\zeta_{s-}(\alpha\sigma)_s(x+\Delta x) + G_{s-}(x)[(\alpha\sigma)_s(x+\Delta x) - (\alpha\sigma)_s(x)]\}dW_s$$

$$-\int_0^t\int_{\mathbf{R}} \{\zeta_{s-}(\alpha\gamma)_s(x+\Delta x) + G_{s-}(x)[(\alpha\gamma)_s(x+\Delta x) - (\alpha\gamma)_s(x)]\}$$

$$\times (\nu(ds,dy) - H(dy)ds).$$

Thus (writing $(\Delta\alpha\sigma)_s = (\alpha\sigma)_s(x+\Delta x) - (\alpha\sigma)_s(x)$ and $(\Delta\alpha\gamma)_s = (\alpha\gamma)_s(x+\Delta x) - (\alpha\gamma)_s(x)$),

$$[\zeta_t(x,\Delta x)]^4 = -4\int_0^t \zeta_{s-}^3\{\zeta_{s-}(\alpha\sigma)_s(x+\Delta x) + G_s(x)(\Delta\alpha\sigma)_s\}dW_s$$

$$-4\int_0^t\int_{\mathbf{R}} \zeta_{s-}^3\{\zeta_{s-}(\alpha\gamma)_s(x+\Delta x) + G_s(x)(\Delta\alpha\gamma)_s\}$$

$$\times (\nu(ds,dy) - H(dy)ds)$$

$$-6\int_0^t\int_{\mathbf{R}} \zeta_{s-}^2[\zeta_{s-}(\alpha\sigma)_s(x+\Delta x) + G_s(x)(\Delta\alpha\sigma)_s]^2 ds$$

$$-\int_0^t\int_{\mathbf{R}} [\zeta_{s-} + (\zeta_{s-}(\alpha\gamma)_s(x+\Delta x) + G_s(x)(\Delta\alpha\gamma)_s)]^4 - \zeta_{s-}^4$$

$$-4\zeta_{s-}^3(\zeta_{s-}(\alpha\gamma)_s(x+\Delta x) + G_s(x)(\Delta\alpha\gamma)_s)\}\nu(ds,dy).$$

Now, truncating as usual by replacing ζ with $F^N(\zeta^N)$ in the above equation (F^N is as in the proof of Lemma 6.6), and then taking expectations, we find that

$$E|\zeta_t^N(x,\Delta x)|^4 \le 12E\int_0^t F^N(\zeta_s^N)^2[F^N(\zeta_s^N)^2\cdot(\alpha\sigma)_s^2(x+\Delta x) + G_s^2(x)(\Delta\alpha\sigma)_s^2]ds$$

$$+12E\int_0^t\int_{\mathbf{R}} F^N(\zeta_s^N)^2[F^m(\zeta_s^N)^2(\alpha\gamma)_s^2(x+\Delta x) + G_s^2(x)(\Delta\alpha\gamma)_s^2]H(dy)ds$$

$$+16E\int_0^t\int_{\mathbf{R}} F^N(\zeta_s^N)[F^N(\zeta_s^N)^3(\alpha\gamma)_s^3(x+\Delta x) + G_s^3(x)(\Delta\gamma)_s^3]H(dy)ds$$

$$+8E\int_0^t\int_{\mathbf{R}} [F^N(\zeta_s^N)^4(\alpha\gamma)_s^4(x+\Delta x) + G_s^4(x)(\Delta\alpha\gamma)_s^4]H(dy)ds.$$

Recall that $|\alpha| \leq \frac{\text{const.}}{|x|}$, and $|\sigma| \leq \text{const.}\ |x|$, so that, by hypothesis, $\alpha\sigma$, $\Delta\alpha\sigma$, $\int_{\mathbf{R}}(\alpha\gamma)^m H(dy)$, and $\int_{\mathbf{R}}(\Delta\alpha\gamma)^m H(dy)$, $m = 1,2,3,4$, are all dominated by a constant C_1 that is independent of x (or Δx). Now, it can be shown that, for any $a \geq 0$, and $0 < \lambda < 1$, we have $a^\lambda \leq \lambda a + 1 - \lambda$.

Thus,

$$E[F^N(\zeta_s^N)^2 G_s^2(x)] \leq (E[F^N(\zeta_s^N)^4])^{1/2} E[G_s^4(x)]^{1/2}$$

$$\leq \frac{1}{2}\{E[F^N(\zeta_s^N)^4] + 1\} E[G_s^4(x)]^{1/2},$$

and

$$E[F^N(\zeta_s^N) G_s^3(x)] \leq (E[F^N(\zeta_s^N)^4])^{1/4} E[G_s^4(x)]^{3/4}$$

$$\leq \frac{1}{4}\{E[F^N(\zeta_s^N)^4] + 3\} E[G_s^4(x)]^{3/4}.$$

However, from Lemma 6.7, $\sup E[G_s^4(x)] \leq C_2$, (which is independent of Δx). Therefore,

$$E[F^N(\zeta_t^N)^4] \leq E[(\zeta_t^N)^4] \leq \text{const.}\Big[\int_0^t E[F^N(\zeta_s^N)^4] + 1\Big],$$

where the constant depends only on C_1, C_2, and T. The result follows from Gronwall's inequality and Fatou's Lemma.

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